AN ELEMENTARY TREATISE

ON

FOURIER'S SERIES

AND

SPHERICAL, CYLINDRICAL, AND ELLIPSOIDAL HARMONICS,

WITH

APPLICATIONS TO PROBLEMS IN MATHEMATICAL PHYSICS.

BY

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PREFACE.

About ten years ago I gave a course of lectures on Trigonometric Series, following closely the treatment of that subject in Riemann's "Partielle Differentialgleichungen," to accompany a short course on The Potential Function, given by Professor B. O. Peirce.

My course has been gradually modified and extended until it has become an introduction to Spherical Harmonics and Bessel's and Lamé's Functions.

Two years ago my lecture notes were lithographed by my class for their own use and were found so convenient that I have prepared them for publication, hoping that they may prove useful to others as well as to my own students. Meanwhile, Professor Peirce has published his lectures on "The Newtonian Potential Function" (Boston, Ginn & Co.), and the two sets of lectures form a course (Math. 10) given regularly at Harvard, and intended as a partial introduction to modern Mathematical Physics.

Students taking this course are supposed to be familiar with so much of the infinitesimal calculus as is contained in my "Differential Calculus" (Boston, Ginn & Co.) and my "Integral Calculus" (second edition, same publishers), to which I refer in the present book as "Dif. Cal." and "Int. Cal." Here, as in the "Calculus," I speak of a "derivative" rather than a "differential coefficient," and use the notation D_x instead of $\frac{\delta}{\delta x}$ for "partial derivative with respect to x."

The course was at first, as I have said, an exposition of Riemann's "Partielle Differentialgleichungen." In extending it, I drew largely from Ferrer's "Spherical Harmonics" and Heine's "Kugelfunctionen," and was somewhat indebted to Todhunter ("Functions of Laplace, Bessel, and Lamé"), Lord Rayleigh ("Theory of Sound"), and Forsyth ("Differential Equations").

In preparing the notes for publication, I have been greatly aided by the criticisms and suggestions of my colleagues, Professor B. O. Peirce and Dr. Maxime Bôcher, and the latter has kindly contributed the brief historical sketch contained in Chapter IX.

W. E. BYERLY.

Cambridge, Mass., Sept. 1893.

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CHAPTER I.

INTRODUCTION.

1. In many important problems in mathematical physics we are obliged to deal with *partial differential equations* of a comparatively simple form.

For example, in the Analytical Theory of Heat we have for the change of temperature of any solid due to the flow of heat within the solid, the equation

$$D_t u = a^2(D_x^2 u + D_y^2 u + D_z^2 u),^* \qquad [\text{I}]$$

where u represents the temperature at any point of the solid and t the time.

In the simplest case, that of a slab of infinite extent with parallel plane faces, where the temperature can be regarded as a function of one coördinate, [I] reduces to

$$D_t u = a^2 D_x^2 u, \qquad [\text{II}]$$

a form of considerable importance in the consideration of the problem of the cooling of the earth's crust.

In the problem of the permanent state of temperatures in a thin rectangular plate, the equation [I] becomes

$$D_x^2 u + D_y^2 u = 0. \qquad [\text{III}]$$

In *polar* or *spherical coördinates* [I] is less simple, it is

$$D_t u = \frac{a^2}{r^2}\left[D_r(r^2 D_r u) + \frac{1}{\sin\theta} D_\theta(\sin\theta\, D_\theta u) + \frac{1}{\sin^2\theta} D_\phi^2 u\right]. \qquad [\text{IV}]$$

In the case where the solid in question is a sphere and the temperature at any point depends merely on the distance of the point from the centre [IV] reduces to

$$D_t(ru) = a^2 D_r^2(ru). \qquad [\text{V}]$$

In *cylindrical coördinates* [I] becomes

$$D_t u = a^2\left[D_r^2 u + \frac{1}{r} D_r u + \frac{1}{r^2} D_\phi^2 u + D_z^2 u\right]. \qquad [\text{VI}]$$

In considering the flow of heat in a cylinder when the temperature at any point depends merely on the distance r of the point from the axis [VI] becomes

$$D_t u = a^2\left(D_r^2 u + \frac{1}{r} D_r u\right). \qquad [\text{VII}]$$

* For the sake of brevity we shall often use the symbol ∇^2 for the operation $D_x^2 + D_y^2 + D_z^2$; and with this notation equation [I] would be written $D_t u = a^2 \nabla^2 u$.

In Acoustics in several problems we have the equation

$$D_t^2 y = a^2 D_x^2 y; \qquad \text{[VIII]}$$

for instance, in considering the transverse or the longitudinal vibrations of a stretched elastic string, or the transmission of plane sound waves through the air.

If in considering the transverse vibrations of a stretched string we take account of the resistance of the air [VIII] is replaced by

$$D_t^2 y + 2k D_t y = a^2 D_x^2 y. \qquad \text{[IX]}$$

In dealing with the vibrations of a stretched elastic membrane, we have the equation

$$D_t^2 z = c^2(D_x^2 z + D_y^2 z), \qquad \text{[X]}$$

or in *cylindrical coördinates*

$$D_t^2 z = c^2\left(D_r^2 z + \frac{1}{r} D_r z + \frac{1}{r^2} D_\phi^2 z\right). \qquad \text{[XI]}$$

In the theory of *Potential* we constantly meet Laplace's Equation

$$D_x^2 V + D_y^2 V + D_z^2 V = 0 \qquad \text{[XII]}$$

or

$$\nabla^2 V = 0$$

which in *spherical coördinates* becomes

$$\frac{1}{r^2}\left[r D_r^2(rV) + \frac{1}{\sin\theta} D_\theta(\sin\theta\, D_\theta V) + \frac{1}{\sin^2\theta} D_\phi^2 V \right] = 0, \qquad \text{[XIII]}$$

and in *cylindrical coördinates*

$$D_r^2 V + \frac{1}{r} D_r V + \frac{1}{r^2} D_\phi^2 V + D_z^2 V = 0. \qquad \text{[XIV]}$$

In *curvilinear coördinates* it is

$$h_1 h_2 h_3 \left[D_{\rho_1}\left(\frac{h_1}{h_2 h_3} D_{\rho_1} V\right) + D_{\rho_2}\left(\frac{h_2}{h_3 h_1} D_{\rho_2} V\right) + D_{\rho_3}\left(\frac{h_3}{h_1 h_2} D_{\rho_3} V\right) \right] = 0; \qquad \text{[XV]}$$

where $\qquad f_1(x,y,z) = \rho_1, \quad f_2(x,y,z) = \rho_2, \quad f_3(x,y,z) = \rho_3$

represent a set of surfaces which cut one another at right angles, no matter what values are given to ρ_1, ρ_2, and ρ_3; and where

$$h_1^2 = (D_x\rho_1)^2 + (D_y\rho_1)^2 + (D_z\rho_1)^2$$

$$h_2^2 = (D_x\rho_2)^2 + (D_y\rho_2)^2 + (D_z\rho_2)^2$$

$$h_3^2 = (D_x\rho_3)^2 + (D_y\rho_3)^2 + (D_z\rho_3)^2,$$

and, of course, must be expressed in terms of ρ_1, ρ_2, and ρ_3.

If it happens that $\nabla^2\rho_1 = 0$, $\nabla^2\rho_2 = 0$, and $\nabla^2\rho_3 = 0$, then Laplace's Equation [XV] assumes the very simple form

$$h_1^2 D_{\rho_1}^2 V + h_2^2 D_{\rho_2}^2 V + h_3^2 D_{\rho_3}^2 V = 0. \qquad \text{[XVI]}$$

2. A *differential equation* is an equation containing derivatives or differentials with or without the primitive variables from which they are derived.

The *general solution* of a differential equation is the equation expressing the most general relation between the primitive variables which is consistent with the given differential equation and which does not involve differentials or derivatives. A general solution will always contain *arbitrary* (*i. e.*, undetermined) *constants* or *arbitrary functions.*

A *particular solution* of a differential equation is a relation between the primitive variables which is consistent with the given differential equation, but which is less general than the general solution, although included in it.

Theoretically, every particular solution can be obtained from the general solution by substituting in the general solution particular values for the arbitrary constants or particular functions for the arbitrary functions; but in practice it is often easy to obtain particular solutions directly from the differential equation when it would be difficult or impossible to obtain the general solution.

3. If a problem requiring for its solution the solving of a differential equation is *determinate*, there must always be given in addition to the differential equation enough outside conditions for the determination of all the arbitrary constants or arbitrary functions that enter into the general solution of the equation; and in dealing with such a problem, if the differential equation can be readily solved the natural method of procedure is to obtain its general solution, and then to determine the constants or functions by the aid of the given conditions.

It often happens, however, that the general solution of the differential equation in question cannot be obtained, and then, since the problem *if determinate* will be solved if by any means a solution of the equation can be found which will also satisfy the given outside conditions, it is worth while to try to get *particular solutions* and so to combine them as to form a result which shall satisfy the given conditions without ceasing to satisfy the differential equation.

4. A differential equation is *linear* when it would be of the first degree if the dependent variable and all its derivatives were regarded as algebraic unknown quantities. If it is linear and contains no term which does not involve the dependent variable or one of its derivatives, it is said to be *linear* and *homogeneous.*

All the differential equations collected in Art. 1 are linear and homogeneous.

5. *If a value of the dependent variable has been found which satisfies a given homogeneous, linear, differential equation, the product formed by multiplying this value by any constant will also be a value of the dependent variable which will satisfy the equation.*

We shall begin by getting a particular solution of [III], and we shall use a device which always succeeds when the equation is *linear* and *homogeneous* and has *constant coefficients*.

Assume * $u = e^{\alpha y + \beta x}$, where α and β are constants, substitute in [III] and divide by $e^{\alpha y + \beta x}$, and we have $\alpha^2 + \beta^2 = 0$. If, then, this condition is satisfied $u = e^{\alpha y + \beta x}$ is a solution.

Hence $u = e^{\alpha y \pm \alpha x i}$ † is a solution of [III], no matter what value may be given to α.

This form is objectionable, since it involves an imaginary. We can, however, readily improve it.

Take $u = e^{\alpha y}\, e^{\alpha x i}$, a solution of [III], and $u = e^{\alpha y}\, e^{-\alpha x i}$, another solution of [III]; add these values of u and divide the sum by 2 and we have $e^{\alpha y} \cos \alpha x$. (v. Int. Cal. Art. 35, [1].) Therefore by Art. 5

$$u = e^{\alpha y} \cos \alpha x \tag{5}$$

is a solution of [III]. Take $u = e^{\alpha y}\, e^{\alpha x i}$ and $u = e^{\alpha y}\, e^{-\alpha x i}$, subtract the second value of u from the first and divide by $2i$ and we have $e^{\alpha y} \sin \alpha x$. (v. Int. Cal. Art. 35, [2]). Therefore by Art. 5

$$u = e^{\alpha y} \sin \alpha x \tag{6}$$

is a solution of [III].

Let us now see if out of these particular solutions we can build up a solution which will satisfy the conditions (1), (2), (3), and (4).

Consider
$$u = e^{\alpha y} \sin \alpha x. \tag{6}$$

It is zero when $x = 0$ for all values of α. It is zero when $x = \pi$ if α is a whole number. It is zero when $y = \infty$ if α is negative. If, then, we write u equal to a sum of terms of the form $Ae^{-my} \sin mx$, where m is a positive integer, we shall have a solution of [III] which satisfies conditions (1), (2) and (3). Let this solution be

$$u = A_1 e^{-y} \sin x + A_2 e^{-2y} \sin 2x + A_3 e^{-3y} \sin 3x + A_4 e^{-4y} \sin 4x + \cdots \tag{7}$$

A_1, A_2, A_3, A_4, &c., being undetermined constants.

When $y = 0$ (7) reduces to

$$u = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + A_4 \sin 4x + \cdots. \tag{8}$$

If now it is possible to develop unity into a series of the form (8), our problem is solved; we have only to substitute the coefficients of that series for A_1, A_2, A_3, &c. in (7).

* This assumption must be regarded as purely tentative. It must be tested by substituting in the equation, and is justified if it leads to a solution.

† We shall regularly use the symbol i for $\sqrt{-1}$.

As in the last problem let* $y = e^{\alpha x + \beta t}$ and substitute in [VIII]. Divide by $e^{\alpha x + \beta t}$ and we have $\beta^2 = a^2\alpha^2$ as the condition that our assumed value of y shall satisfy the equation.

$$y = e^{\alpha x \pm a\alpha t} \tag{5}$$

is, then, a solution of (VIII) whatever the value of α.

It is more convenient to have a trigonometric than an exponential form to deal with, and we can readily obtain one by using an imaginary value for α in (5). Replace α by αi and (5) becomes $y = e^{(x \pm at)\alpha i}$, a solution of [VIII]. Replace α by $-\alpha i$ and (5) becomes $y = e^{-(x \pm at)\alpha i}$, another solution of [VIII]. Add these values of y and divide by 2 and we have $\cos \alpha(x \pm at)$. Subtract the second value of y from the first and divide by $2i$ and we have $\sin \alpha(x \pm at)$.

$$y = \cos \alpha(x + at)$$
$$y = \cos \alpha(x - at)$$
$$y = \sin \alpha(x + at)$$
$$y = \sin \alpha(x - at)$$

are, then, solutions of [VIII]. Writing y successively equal to half the sum of the first pair of values, half their difference, half the sum of the last pair of values, and half their difference, we get the very convenient particular solutions of [VIII].

$$y = \cos \alpha x \cos a\alpha t$$
$$y = \sin \alpha x \sin a\alpha t$$
$$y = \sin \alpha x \cos a\alpha t$$
$$y = \cos \alpha x \sin a\alpha t.$$

If we take the third form

$$y = \sin \alpha x \cos a\alpha t$$

it will satisfy conditions (1) and (4), no matter what value may be given to α, and it will satisfy (2) if $\alpha = \frac{m\pi}{l}$ where m is an integer.

If then we take

$$y = A_1 \sin \frac{\pi x}{l} \cos \frac{\pi a t}{l} + A_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi a t}{l} + A_3 \sin \frac{3\pi x}{l} \cos \frac{3\pi a t}{l} + \cdots \tag{6}$$

where A_1, A_2, A_3 $\cdots$ are undetermined constants, we shall have a solution of [VIII] which satisfies (1), (2), and (4). When $t = 0$ it reduces to

$$y = A_1 \sin \frac{\pi x}{l} + A_2 \sin \frac{2\pi x}{l} + A_3 \sin \frac{3\pi x}{l} + \cdots \tag{7}$$

If now it is possible to develop $f(x)$ into a series of the form (7), we can solve our problem completely. We have only to take the coefficients of this series as values of A_1, A_2, A_3 . . . in (6), and we shall have a solution of [VIII] which satisfies all our given conditions.

* See note on page 5.

Divide by r^m and use the notation of ordinary derivatives since P depends upon θ only, and we have the equation

$$m(m+1)P + \frac{1}{\sin\theta}\frac{d\left(\sin\theta\frac{dP}{d\theta}\right)}{d\theta} = 0, \tag{3}$$

from which to obtain P.

Equation (3) can be simplified by changing the independent variable. Let $x = \cos\theta$ and (3) becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + m(m+1)P = 0. \tag{4}$$

Assume * now that P can be expressed as a sum or as a series of terms involving whole powers of x multiplied by constant coefficients.

Let $P = \Sigma a_n x^n$ and substitute this value of P in (4). We get

$$\Sigma\left[n(n-1)a_n x^{n-2} - n(n+1)a_n x^n + m(m+1)a_n x^n\right] = 0, \tag{5}$$

where the symbol Σ indicates that we are to form all the terms we can by taking successive whole numbers for n.

As (5) must be true no matter what the value of x, the coefficient of any given power of x, as for instance x^k, must vanish. Hence

$$(k+2)(k+1)a_{k+2} - k(k+1)a_k + m(m+1)a_k = 0 \tag{6}$$

and

$$a_{k+2} = -\frac{m(m+1)-k(k+1)}{(k+1)(k+2)}a_k. \tag{7}$$

If now any set of coefficients satisfying the relation (7) be taken, $P = \Sigma a_k x^k$ will be a solution of (4).

If $\qquad k = m, \quad a_{k+2} = 0, \quad a_{k+4} = 0, \quad$ &c.

Since it will answer our purpose if we pick out the simplest set of coefficients that will obey the condition (7), we can take a set including a_m.

Let us rewrite (7) in the form

$$a_k = -\frac{(k+2)(k+1)}{(m-k)(m+k+1)}a_{k+2}. \tag{8}$$

We get from (8), beginning with $k = m-2$,

$$a_{m-2} = -\frac{m(m-1)}{2.(2m-1)}a_m$$

$$a_{m-4} = \frac{m(m-1)(m-2)(m-3)}{2.4.(2m-1)(2m-3)}a_m$$

$$a_{m-6} = -\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{2.4.6.(2m-1)(2m-3)(2m-5)}a_m, \quad \text{\&c}$$

* See note on page 5.

If m is even we see that the set will end with a_0, if m is odd, with a_1.

$$P = a_m\left[x^m - \frac{m(m-1)}{2.(2m-1)}x^{m-2} + \frac{m(m-1)(m-2)(m-3)}{2.4.(2m-1)(2m-3)}x^{m-4} - \ldots\right]$$

where a_m is entirely arbitrary, is, then, a solution of (4). It is found convenient to take a_m equal to

$$\frac{(2m-1)(2m-3)\cdots 1}{m!}$$

and it can be shown that with this value of a_m $P=1$ when $x=1$.

P is a function of x and contains no higher powers of x than x^m. It is usual to write it as $P_m(x)$.

We proceed to compute a few values of $P_m(x)$ from the formula

$$P_m(x) = \frac{(2m-1)(2m-3)\cdots 1}{m!}\left[x^m - \frac{m(m-1)}{2.(2m-1)}x^{m-2} + \frac{m(m-1)(m-2)(m-3)}{2.4.(2m-1)(2m-3)}x^{m-4} - \ldots\right]. \qquad (9)$$

We have:

$$\left.\begin{array}{lll}
P_0(x) = 1 & \text{or} & P_0(\cos\theta) = 1 \\
P_1(x) = x & \text{"} & P_1(\cos\theta) = \cos\theta \\
P_2(x) = \frac{1}{2}(3x^2-1) & \text{"} & P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1) \\
P_3(x) = \frac{1}{2}(5x^3-3x) & \text{"} & P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta) \\
P_4(x) = \frac{1}{8}(35x^4-30x^2+3) & \text{or} & \\
 & & P_4(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3) \\
P_5(x) = \frac{1}{8}(63x^5-70x^3+15x) & \text{or} & \\
 & & P_5(\cos\theta) = \frac{1}{8}(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta).
\end{array}\right\} \qquad (10)$$

We have obtained $P = P_m(x)$ as a particular solution of (4) and $P = P_m(\cos\theta)$ as a particular solution of (3). $P_m(x)$ or $P_m(\cos\theta)$ is a new function, known as a *Legendre's Coefficient*, or as a *Surface Zonal Harmonic*, and occurs as a normal form in many important problems.

$V = r^m P_m(\cos\theta)$ is a particular solution of (2) and $r^m P_m(\cos\theta)$ is sometimes called a *Solid Zonal Harmonic*.

We can now proceed to the solution of our original problem

$$V = A_0 r^0 P_0(\cos\theta) + A_1 r P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta) + A_3 r^3 P_3(\cos\theta) + \ldots \qquad (11)$$

where A_0, A_1, A_2, &c., are entirely arbitrary, is a solution of (2) (v. Art. 5).

When $\theta = 0$ (11) reduces to

$$V = A_0 + A_1 r + A_2 r^2 + A_3 r^3 + \ldots,$$

since, as we have said, $P_m(x) = 1$ when $x = 1$, or $P_m(\cos\theta) = 1$ when $\theta = 0$.

By our condition (1)

$$V = \frac{M}{(c^2+r^2)^{\frac{1}{2}}}$$

when $\theta = 0$.

By the Binomial Theorem

$$\frac{M}{(c^2+r^2)^{\frac{1}{2}}}=\frac{M}{c}\left[1-\frac{1}{2}\frac{r^2}{c^2}+\frac{1.3}{2.4}\frac{r^4}{c^4}-\frac{1.3.5}{2.4.6}\frac{r^6}{c^6}+\cdots\right]$$

provided $r<c$. Hence

$$V=\frac{M}{c}\left[P_0(\cos\theta)-\frac{1}{2}\frac{r^2}{c^2}P_2(\cos\theta)+\frac{1.3}{2.4}\frac{r^4}{c^4}P_4(\cos\theta)-\frac{1.3.5}{2.4.6}\frac{r^6}{c^6}P_6(\cos\theta)+\cdots\right] \quad (12)$$

is our required solution if $r<c$; for it is a solution of equation (2) and satisfies condition (1).

EXAMPLE.

Taking the mass of the ring as one pound and the radius of the ring as one foot, compute to two decimal places the value of the potential function due to the ring at the points

(a) $(r=.2,\ \theta=0)$; (d) $(r=.6,\ \theta=0)$; (f) $\left(r=.6,\ \theta=\frac{\pi}{3}\right)$;

(b) $\left(r=.2,\ \theta=\frac{\pi}{4}\right)$; (e) $\left(r=.6,\ \theta=\frac{\pi}{6}\right)$; (g) $\left(r=.6,\ \theta=\frac{\pi}{2}\right)$;

(c) $\left(r=.2,\ \theta=\frac{\pi}{2}\right)$;

Ans. (a) .98; (b) .99; (c) 1.01; (d) .86; (e) .90; (f) 1.00; (g) 1.10.

The unit used is the potential due to a pound of mass concentrated at a point and attracting a second pound of mass concentrated at a point, the two points being a foot apart.

10. A slightly different problem calling for development in terms of Zonal Harmonics is the following:

Required the permanent temperatures within a solid sphere of radius 1, one half of the surface being kept at the constant temperature zero, and the other half at the constant temperature unity.

Let us take the diameter perpendicular to the plane separating the unequally heated surfaces as our axis and let us use spherical coördinates. As in the last problem, we must solve the equation

$$rD_r^2(ru)+\frac{1}{\sin\theta}D_\theta(\sin\theta\, D_\theta u)+\frac{1}{\sin^2\theta}D_\phi^2 u=0 \qquad \text{[XIII] Art. 1}$$

which as before reduces to

$$rD_r^2(ru)+\frac{1}{\sin\theta}D_\theta(\sin\theta\, D_\theta u)=0 \qquad (1)$$

from the consideration that the temperatures must be independent of ϕ.

Our equation of condition is

$$u=1 \text{ from } \theta=0 \text{ to } \theta=\frac{\pi}{2} \text{ and } u=0 \text{ from } \theta=\frac{\pi}{2} \text{ to } \theta=\pi, \qquad (2)$$

when $r=1$.

As we have seen $u = r^m P_m(\cos\theta)$ is a particular solution of (1), m being any positive whole number, and

$$u = A_0 r^0 P_0(\cos\theta) + A_1 r P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta) + A_3 r^3 P_3(\cos\theta) + \cdots \quad (3)$$

where $A_0, A_1, A_2, A_3 \cdots$ are undetermined constants, is a solution of (1).

When $r = 1$ (3) reduces to

$$u = A_0 P_0(\cos\theta) + A_1 P_1(\cos\theta) + A_2 P_2(\cos\theta) + A_3 P_3(\cos\theta) + \cdots \quad (4)$$

If then we can develop our function of θ which enters into equation (2) in a series of the form (4), we have only to take the coefficients of that series as the values of A_0, A_1, A_2, &c., in (3) and we shall have our required solution.

11. As a last example we shall take the problem of the vibration of a stretched circular membrane fastened at the circumference, that is, of an ordinary drum head. We shall suppose the membrane initially distorted into any given form which has circular symmetry * about an axis through the centre perpendicular to the plane of the boundary, and then allowed to vibrate.

Here we have to solve

$$D_t^2 z = c^2\left(D_r^2 z + \frac{1}{r} D_r z + \frac{1}{r^2} D_\phi^2 z\right) \qquad \text{[XI] Art. 1}$$

subject to the conditions

$$z = f(r) \text{ when } t = 0 \quad (1)$$
$$D_t z = 0 \quad \text{“} \quad t = 0 \quad (2)$$
$$z = 0 \quad \text{“} \quad r = a. \quad (3)$$

From the symmetry of the supposed initial distortion z must be independent of ϕ, therefore [XI] reduces to

$$D_t^2 z = c^2\left(D_r^2 z + \frac{1}{r} D_r z\right) \quad (4)$$

and this is the equation for which we wish to find a particular solution.

We shall employ a device not unlike that used in Art. 9.

Assume † $z = R.T$ where R is a function of r alone and T is a function of t alone. Substitute this value of z in (4) and we get

$$R D_t^2 T = c^2 T\left(D_r^2 R + \frac{1}{r} D_r R\right)$$

or

$$\frac{1}{c^2 T}\frac{d^2 T}{dt^2} = \frac{1}{R}\left(\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right). \quad (5)$$

The second member of (5) does not involve t, therefore its equal the first member must be independent of t. The first member of (5) does not involve

* A function of the coördinates of a point has *circular symmetry* about an axis when its value is not affected by rotating the point through any angle about the axis. A surface has circular symmetry about an axis when it is a surface of revolution about the axis.

† See note on page 5.

r, and consequently since it contains neither t nor r, it must be constant. Let it equal $-\mu^2$, where μ of course is an undetermined constant.

Then (5) breaks up into the two differential equations

$$\frac{d^2T}{dt^2} + \mu^2c^2T = 0 \tag{6}$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \mu^2R = 0. \tag{7}$$

(6) can be solved by familiar methods, and we get $T = \cos \mu ct$ and $T = \sin \mu ct$ as simple particular solutions (v. Int. Cal. p. 319, § 21).

To solve (7) is not so easy. We shall first simplify it by a change of independent variable. Let $r = \frac{x}{\mu}$. (7) becomes

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + R = 0. \tag{8}$$

Assume, as in Art. 9, that R can be expressed in terms of whole powers of x. Let $R = \Sigma a_n x^n$ and substitute in (8). We get

$$\Sigma\left[n(n-1)a_n x^{n-2} + na_n x^{n-2} + a_n x^n\right] = 0,$$

an equation which must be true no matter what the value of x. The coefficient of any given power of x, as x^{k-2}, must, then, vanish, and

$$k(k-1)a_k + ka_k + a_{k-2} = 0$$

or

$$k^2a_k + a_{k-2} = 0$$

whence we obtain

$$a_{k-2} = -k^2a_k \tag{9}$$

as the only relation that need be satisfied by the coefficients in order that $R = \Sigma a_k x^k$ shall be a solution of (8).

If $\quad k = 0, \quad a_{k-2} = 0, \quad a_{k-4} = 0, \quad$ &c.

We can then begin with $k = 0$ as our lowest subscript.

From (9)

$$a_k = -\frac{a_{k-2}}{k^2}.$$

Then

$$a_2 = -\frac{a_0}{2^2}$$

$$a_4 = \frac{a_0}{2^2.4^2}$$

$$a_6 = -\frac{a_0}{2^2.4^2.6^2}, \text{ \&c.}$$

Hence

$$R = a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots\right]$$

where a_0 may be taken at pleasure, is a solution of (8), provided the series is convergent.

Take $a_0=1$, and then $R=J_0(x)$ where

$$J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\frac{x^8}{2^2.4^2.6^2.8^2}-\cdots \qquad (10)$$

is a solution of (8).

$J_0(x)$ is easily shown to be convergent for all values real or imaginary of x, since the series made up of the moduli of the terms of $J_0(x)$ (v. Int. Cal. Art. 30)

$$1+\frac{r^2}{2^2}+\frac{r^4}{2^2.4^2}+\frac{r^6}{2^2.4^2.6^2}+\cdots,$$

where r is the modulus of x, is convergent for all values of r. For the ratio of the $n+1$ st term of this series to the n th term is $\frac{r^2}{4n^2}$ and approaches zero as its limit as n is indefinitely increased, no matter what the value of r. Therefore $J_0(x)$ is *absolutely convergent.*

$J_0(x)$ is a new and important form. It is called a *Bessel's Function* of the zero th order, or a *Cylindrical Harmonic.*

Equation (8) was obtained from (7) by the substitution of $x=\mu r$, therefore

$$R=J_0(\mu r)=1-\frac{(\mu r)^2}{2^2}+\frac{(\mu r)^4}{2^2.4^2}-\frac{(\mu r)^6}{2^2.4^2.6^2}+\cdots$$

is a solution of (7), no matter what the value of μ, and $z=J_0(\mu r)\cos\mu ct$ or $z=J_0(\mu r)\sin\mu ct$ is a solution of (4).

$z=J_0(\mu r)\cos\mu ct$ satisfies condition (2) whatever the value of μ. In order that it should also satisfy condition (3) μ must be so taken that

$$J_0(\mu a)=0\,; \qquad (11)$$

that is, μ must be a root of (11) regarded as an equation in μ.

It can be shown that $J_0(x)=0$ has an infinite number of real positive roots, any one of which can be obtained to any required degree of approximation without serious difficulty. Let $x_1, x_2, x_3, \cdots$ be these roots. Then if

$$\frac{x_1}{a}=\mu_1,\ \frac{x_2}{a}=\mu_2,\ \frac{x_3}{a}=\mu_3,\ \&c.,$$

$$z=A_1J_0(\mu_1 r)\cos\mu_1 ct+A_2J_0(\mu_2 r)\cos\mu_2 ct+A_3J_0(\mu_3 r)\cos\mu_3 ct+\cdots \qquad (12)$$

where A_1, A_2, A_3, &c., are any constants, is a solution of (4) which satisfies conditions (2) and (3).

When $t=0$ (12) reduces to

$$z=A_1J_0(\mu_1 r)+A_2J_0(\mu_2 r)+A_3J_0(\mu_3 r)+\cdots. \qquad (13)$$

If then $f(r)$ can be expressed as a series of the form just given, the solution of our problem can be obtained by substituting the coefficients of that series for A_1, A_2, A_3, &c., in (12).

EXAMPLE.

The temperature of a long cylinder is at first unity throughout. The convex surface is then kept at the constant temperature zero. Show that the temperature of any point in the cylinder at the expiration of the time t is

$$u = A_1 e^{-a^2\mu_1^2 t} J_0(\mu_1 r) + A_2 e^{-a^2\mu_2^2 t} J_0(\mu_2 r) + A_3 e^{-a^2\mu_3^2 t} J_0(\mu_3 r) + \cdots$$

where μ_1, μ_2, &c., are the roots of $J_0(\mu c) = 0$, and where

$$1 = A_1 J_0(\mu_1 r) + A_2 J_0(\mu_2 r) + A_3 J_0(\mu_3 r) + \cdots,$$

c being the radius of the cylinder.

12. Each of the five problems which we have taken up forces upon us the consideration of the development of a given function in terms of some *normal form*, and in two of them the normal form suggested is an unfamiliar function. It is clear, then, that a complete treatment of our subject will require the investigation of the properties and relations of certain new and important functions, as well as the consideration of methods of developing in terms of them.

13. In each of the problems just taken up we have to deal with a homogeneous linear partial differential equation involving two independent variables, and we are content if we can obtain particular solutions. In each case the assumption made in the last problem, that there exists a solution of the equation in which the dependent variable is the product of two factors each of which involves but one of the independent variables, will reduce the question to solving two ordinary differential equations which can be treated separately.

If these equations are familiar ones their solutions can be written down at once; if unfamiliar, the device used in problems 3 and 5 is often serviceable, namely, that of assuming that the dependent variable can be expressed as a sum or series of terms involving whole powers of the independent variable, and then determining the coefficients.

Let us consider again the equations used in the first, second and third problems.

$$(a) \qquad D_x^2 u + D_y^2 u = 0 \tag{1}$$

Assume $u = X.Y$ where X involves x but not y, and Y involves y but not x.

Substitute in (1), $\qquad Y D_x^2 X + X D_y^2 Y = 0,$

or, since we are now dealing with functions of a single variable,

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0,$$

or

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -\frac{1}{X}\frac{d^2X}{dx^2}. \tag{2}$$

Since by the reasoning used in (a) and (b) each member of (2) must be a constant, say a^2, we have

$$r\frac{d^2(rR)}{dr^2} = a^2R \tag{3}$$

and

$$\frac{1}{\sin\theta}\frac{d\left(\sin\theta\frac{d\Theta}{d\theta}\right)}{d\theta} + a^2\Theta = 0. \tag{4}$$

(3) can be expanded into

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - a^2R = 0\,. \tag{5}$$

(5) can be solved (v. Int. Cal. p. 321, § 23), and has for its complete solution

$$R = Ar^m + Br^n\,,$$

where $\quad m = -\tfrac{1}{2} + \sqrt{a^2 + \tfrac{1}{4}} \quad$ and $\quad n = -\tfrac{1}{2} - \sqrt{a^2 + \tfrac{1}{4}}\,.$

Hence $n = -m - 1$, and a^2 may be written $m(m+1)$, m being wholly arbitrary; and

$$R = Ar^m + Br^{-m-1}\,.$$

$$R = r^m\,, \qquad \text{and} \qquad R = \frac{1}{r^{m+1}}$$

are, then, particular solutions of

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - m(m+1)R = 0\,. \tag{6}$$

With the new value of a^2 (4) becomes

$$\frac{1}{\sin\theta}\frac{d\left(\sin\theta\frac{d\Theta}{d\theta}\right)}{d\theta} + m(m+1)\Theta = 0\,. \tag{7}$$

which has been treated in Art. 9 for the case where m is a positive integer, and the particular solution $\Theta = P_m(\cos\theta)$ has been obtained.

Hence $$V = r^m P_m(\cos\theta)$$

and $$V = \frac{1}{r^{m+1}}P_m(\cos\theta)\,,$$

m being a positive integer, are particular solutions of (1). The first of these was obtained in Art. 9, but the second is new and exceedingly important.

14. The method of obtaining a particular solution of an ordinary linear differential equation, which we have used in Articles 9 and 11, is of very extensive application, and often leads to the general solution of the equation in question.

As a very simple example, let us take the equation Art. 13 (a) (4), which we shall write

$$\frac{d^2z}{dx^2} + a^2z = 0 . \tag{1}$$

Assume that there is a solution which can be expressed in terms of powers of x; that is, let $z = \Sigma\, a_n x^n$, where the coefficients are to be determined. Substitute this value for z in (1) and we get

$$\Sigma\left[n(n-1)a_n x^{n-2} + a^2 a_n x^n\right] = 0 .$$

Since this equation must be true from its form, without reference to the value of x, that is, since it must be an identical equation, the coefficient of each power of x must equal zero, and we have

$$(n+1)(n+2)a_{n+2} + a^2 a_n = 0 ;$$

whence

$$a_n = -\frac{(n+1)(n+2)}{a^2} a_{n+2}$$

is the only relation that need hold between the coefficients in order that $z = \Sigma\, a_n x^n$ should be a solution of (1).

If $n+2=0$ or $n+1=0$, a_n will be zero and a_{n-2}, a_{n-4}, &c., will be zero. In the first case the series will begin with a_0, in the second with a_1.

$$a_{n+2} = -\frac{a^2}{(n+1)(n+2)} a_n .$$

If we begin with a_0 we have

$$a_2 = -\frac{a^2}{2!} a_0 , \qquad a_4 = \frac{a^4}{4!} a_0 , \qquad a_6 = -\frac{a^6}{6!} a_0 , \text{ \&c.}, \ldots$$

and

$$z = a_0\left(1 - \frac{a^2x^2}{2!} + \frac{a^4x^4}{4!} - \frac{a^6x^6}{6!} + \cdots\right) \tag{2}$$

or

$$z = a_0 \cos ax \tag{3}$$

is a particular solution of (1).

If we begin with a_1 we have

$$a_3 = -\frac{a^2}{3!} a_1 , \qquad a_5 = \frac{a^4}{5!} a_1 , \qquad a_7 = -\frac{a^6}{7!} a_1 , \text{ \&c.}, \ldots$$

and

$$z = a_1\left(x - \frac{a^2x^3}{3!} + \frac{a^4x^5}{5!} - \frac{a^6x^7}{7!} + \cdots\right) \tag{4}$$

is a solution of (1); a_1 can be taken at pleasure. Let $a_1 = a$, (4) becomes

$$z = ax - \frac{a^3x^3}{3!} + \frac{a^5x^5}{5!} - \frac{a^7x^7}{7!} + \cdots$$

or $$z = \sin ax$$

which, then, is a particular solution of (1).

$$z = A \sin ax + B \cos ax \tag{5}$$

is, then, a solution of (1), and since it contains two arbitrary constants it is the general solution.

15. As another example we will take the equation

$$x^2\frac{d^2z}{dx^2} + 2x\frac{dz}{dx} - m(m+1)z = 0\,, \tag{1}$$

which is in effect equation (6), Art. 13 (c), and let m be a positive integer.

Assume $z = \Sigma a_n x^n$ and substitute in (1). We get

$$\Sigma\,[n(n+1) - m(m+1)]\,a_n x^n = 0\,.$$

This is an identical equation, therefore

$$[n(n+1) - m(m+1)]a_n = 0\,.$$

Hence $a_n = 0$ for all values of n except those which make

$$n(n+1) - m(m+1) = 0\,,$$

that is, for all values of n except $n = m$ and $n = -m-1$. Then

$$z = Ax^m + Bx^{-m-1} \tag{2}$$

is the general solution of (1) and

$$z = x^m \quad \text{and} \quad z = \frac{1}{x^{m+1}}$$

are particular solutions. If m is not a positive integer this method will not lead to a result, and we are driven back to that employed in Art. 13 (c).

16. Let us now take the equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dz}{dx}\right] + m(m+1)z = 0 \tag{1}$$

which is in effect equation (4), Art. 9, and is known as *Legendre's Equation*. (1) may be written

$$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + m(m+1)z = 0\,. \tag{2}$$

If m is a positive even whole number, $p_m(x)$ will terminate with the term containing x^m, and is easily seen to be identical with

$$(-1)^{\frac{m}{2}} \frac{2^m \left[\Gamma\left(\frac{m}{2}+1\right)\right]^2}{\Gamma(m+1)} P_m(x). \qquad \text{[v. Art. 9 (9)]}$$

For all other values of m, $p_m(x)$ is a series.

The ratio of the $(n+1)$st term of $p_m(x)$ to the nth, when m is not a positive even integer, is

$$\frac{(2n-2-m)(2n-1+m)}{(2n-1)(2n)} x^2.$$

Its limiting value, as n is increased, is x^2, and the series is therefore convergent if $-1 < x < 1$. It is divergent for all other values of x.

If m is a positive odd whole number $q_m(x)$ will terminate with the term containing x^m, and is easily seen to be identical with

$$(-1)^{\frac{m-1}{2}} \frac{2^{m-1}\left[\Gamma\left(\frac{m+1}{2}\right)\right]^2}{\Gamma(m+1)} P_m(x).$$

For all other values of m, $q_m(x)$ is a series, and can be shown to be convergent if $-1 < x < 1$, and divergent for all other values of x.

$$z = A p_m(x) + B q_m(x) \tag{6}$$

is the general solution of Legendre's Equation if $-1 < x < 1$, no matter what the value of m. From Art. 13 (c) it follows that

$$\left.\begin{aligned} V &= r^m p_m(\cos\theta) \\ V &= \frac{1}{r^{m+1}} p_m(\cos\theta) \\ V &= r^m q_m(\cos\theta) \\ V &= \frac{1}{r^{m+1}} q_m(\cos\theta) \end{aligned}\right\} \tag{7}$$

are particular solutions of

$$r D_r^2(rV) + \frac{1}{\sin\theta} D_\theta(\sin\theta\, D_\theta V) = 0,$$

no matter what the value of m, provided $\cos\theta$ is neither one nor minus one.

In the work we shall have to do with Laplace's and Legendre's Equations, it is generally possible to restrict m to being a positive integer, and hereafter we shall usually confine our attention to that case.

With this understanding let us return to (3), which may be rewritten

$$a_{n+2} = -\frac{(m-n)(m+n+1)}{(n+1)(n+2)} a_n.$$

If $\qquad a_{n+2} = 0$, then $a_{n+4} = 0$, $a_{n+6} = 0$, &c.;

but $\qquad a_{n+2} = 0$ if $n = m$, or $n = -m-1$.

If in (3) we begin with $n = m - 2$, we get the sequence of coefficients already obtained in Art. 9, and we have $z = P_m(x)$, where

$$P_m(x) = \frac{(2m-1)(2m-3)\cdots 1}{m!}\left[x^m - \frac{m(m-1)}{2(2m-1)}x^{m-2}\right.$$

$$+\frac{m(m-1)(m-2)(m-3)}{2.4.(2m-1)(2m-3)}x^{m-4}$$

$$\left.-\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{2.4.6.(2m-1)(2m-3)(2m-5)}x^{m-6}+\cdots\right], \tag{8}$$

as a particular solution of Legendre's Equation.

If, however, we begin with $n = -m-3$, we have

$$a_{-m-3} = \frac{(m+1)(m+2)}{2(2m+3)} a_{-m-1}$$

$$a_{-m-5} = \frac{(m+1)(m+2)(m+3)(m+4)}{2.4.(2m+3)(2m+5)} a_{-m-1}$$

$$a_{-m-7} = \frac{(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)}{2.4.6.(2m+3)(2m+5)(2m+7)} a_{-m-1}, \text{ \&c.}$$

a_{-m-1} may be taken at pleasure, and is usually taken as $\dfrac{m!}{1.3.5\cdots(2m+1)}$, and $z = Q_m(x)$ where

$$Q_m(x) = \frac{m!}{(2m+1)(2m-1)\cdots 1}\left[\frac{1}{x^{m+1}} + \frac{(m+1)(m+2)}{2.(2m+3)}\frac{1}{x^{m+3}}\right.$$

$$\left.+\frac{(m+1)(m+2)(m+3)(m+4)}{2.4.(2m+3)(2m+5)}\frac{1}{x^{m+5}}+\cdots\right] \tag{9}$$

is a second particular solution of Legendre's Equation, provided the series is convergent. $Q_m(x)$ is called a *Surface Zonal Harmonic of the second kind.*

It is easily seen to be convergent if $x < -1$ or $x > 1$, and divergent if $-1 < x < 1$.

Hence if m is a positive integer,

$$z = A\,P_m(x) + B\,Q_m(x) \tag{10}$$

is the general solution of Legendre's Equation if $x < -1$ or $x > 1$.

We have seen that for $-1 < x < 1$

$$P_m(x) = (-1)^{\frac{m}{2}} \frac{\Gamma(m+1)}{2^m\left[\Gamma\left(\frac{m}{2}+1\right)\right]^2}\,p_m(x) \tag{11}$$

if m is an even integer, and

$$P_m(x) = (-1)^{\frac{m-1}{2}} \frac{\Gamma(m+1)}{2^{m-1}\left[\Gamma\left(\frac{m+1}{2}\right)\right]^2}\,q_m(x) \tag{12}$$

if m is an odd integer.

If now we define $Q_m(x)$ as follows when $-1 < x < 1$

$$Q_m(x) = (-1)^{\frac{m+1}{2}} \frac{2^{m-1}\left[\Gamma\left(\frac{m+1}{2}\right)\right]^2}{\Gamma(m+1)}\,p_m(x) \tag{13}$$

if m is an odd integer, and

$$Q_m(x) = (-1)^{\frac{m}{2}} \frac{2^m\left[\Gamma\left(\frac{m}{2}+1\right)\right]^2}{\Gamma(m+1)}\,q_m(x) \tag{14}$$

if m is an even integer, then (10) will be the general solution of Legendre's Equation if m is a positive integer when $-1 < x < 1$, as well as when $x < -1$ or $x > 1$.

17. Let us last consider the equation

$$\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx} + \left(1 - \frac{m^2}{x^2}\right)z = 0 \tag{1}$$

which is known as Bessel's Equation, and which reduces to (8) Art. 11, that is, to

$$\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx} + z = 0$$

when $m = 0$;* (1) can be simplified by a change of the dependent variable.

* This equation was first studied by Fourier in considering the cooling of a cylinder. We shall designate it as "Fourier's Equation."

Let $z = x^m v$ and we get

$$\frac{d^2v}{dx^2} + \frac{2m+1}{x}\frac{dv}{dx} + v = 0 \tag{2}$$

to determine v.

Assume $v = \Sigma a_n x^n$, and substitute in (2). We get

$$\Sigma\left[n(2m+n)a_n x^{n-2} + a_n x^n\right] = 0;$$

whence $$a_{n-2} = -n(2m+n)a_n.$$

If we begin with $n = 0$, then $a_{n-2} = 0$, $a_{n-4} = 0$, &c., and we have the set of values

$$a_2 = -\frac{a_0}{2(2m+2)} = -\frac{a_0}{2^2(m+1)}$$

$$a_4 = \frac{a_0}{2.4(2m+2)(2m+4)} = \frac{a_0}{2^4.2!(m+1)(m+2)}$$

$$a_6 = -\frac{a_0}{2.4.6(2m+2)(2m+4)(2m+6)} = -\frac{a_0}{2^6.3!(m+1)(m+2)(m+3)};$$

whence $$z = a_0x^m\left[1 - \frac{x^2}{2^2(m+1)} + \frac{x^4}{2^4.2!(m+1)(m+2)}\right.$$

$$\left. - \frac{x^6}{2^6.3!(m+1)(m+2)(m+3)} + \cdots\right] \tag{3}$$

is a solution of Bessel's Equation. a_0 is usually taken as $\frac{1}{2^m m!}$, if m is a positive integer, or as $\frac{1}{2^m\Gamma(m+1)}$ if m is unrestricted in value, and the second member of (3) is represented by $J_m(x)$ and is called a *Bessel's Function* of the mth order, or a *Cylindrical Harmonic* of the mth order.

If $m = 0$, $J_m(x)$ becomes $J_0(x)$ and is the value of z obtained in Art. 11 as the solution of equation (8) of that article.

If in equation (1) we substitute $x^{-m}v$ in place of $x^m v$ for z, we get in place of (2) the equation

$$\frac{d^2v}{dx^2} + \frac{1-2m}{x}\frac{dv}{dx} + v = 0$$

and in place of (3)

$$z = a_0x^{-m}\left[1 - \frac{x^2}{2^2(1-m)} + \frac{x^4}{2^4.2!(1-m)(2-m)}\right.$$

$$\left. - \frac{x^6}{2^6.3!(1-m)(2-m)(3-m)} + \cdots\right] \tag{4}$$

If a_0 is taken equal to $\dfrac{1}{2^{-m}\Gamma(1-m)}$ the second member of (4) is the same function of $-m$ and x that $J_m(x)$ is of $+m$ and x and may be written $J_{-m}(x)$.

Therefore
$$z = AJ_m(x) + BJ_{-m}(x) \tag{5}$$

is the general solution of (1) unless $J_m(x)$ and $J_{-m}(x)$ should prove not to be independent.

It is easily seen that when $m = 0$, $J_{-m}(x)$ and $J_m(x)$ become identical and (5) reduces to

$$z = (A + B)J_0(x)$$

and contains but a single arbitrary constant and is not the general solution of Fourier's Equation (8) Art. (11).

It can be shown that $J_{-m}(x) = (-1)^m J_m(x)$ whenever m is an integer, and consequently that the solution (5) is general only when m if real is fractional or incommensurable.

The general solution for the important case where $m = 0$ is, however, easily obtained. Let $F(m,x)$ be the value which the second member of (3) assumes when $a_0 = 1$; then the value which the second member of (4) assumes when $a_0 = 1$ will be $F(-m,x)$, and it has been shown that $z = F(m,x)$ and $z = F(-m,x)$ are solutions of Bessel's Equation; $z = F(m,x) - F(-m,x)$ is, then, a solution, as is also

$$z = \frac{F(m,x) - F(-m,x)}{2m}, \tag{6}$$

but the limiting value which $\dfrac{F(m,x) - F(-m,x)}{2m}$ approaches as m approaches zero is $[D_m F(m,x)]_{m=0}$ and consequently

$$z = [D_m F(m,x)]_{m=0} \tag{7}$$

is a solution of the equation

$$\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx} + z = 0, \tag{8}$$

and the general solution of (8) is

$$z = AJ_0(x) + B[D_m F(m,x)]_{m=0}.$$

$$F(m,x) = x^m\left[1 - \frac{x^2}{2^2(m+1)} + \frac{x^4}{2^4.2!(m+1)(m+2)} - \frac{x^6}{2^6.3!(m+1)(m+2)(m+3)} + \cdots\right]$$

$$D_m F(m,x) = x^m \log x \left[1 - \frac{x^2}{2^2(m+1)} + \frac{x^4}{2^4.2!(m+1)(m+2)} - \cdots\right]$$
$$+ x^m D_m \left[1 - \frac{x^2}{2^2(m+1)} + \frac{x^4}{2^4.2!(m+1)(m+2)} + \cdots\right].$$

The general term of the last parenthesis can be written

$$(-1)^k \frac{x^{2k}}{2^{2k}.\,k!(m+1)(m+2)\cdots(m+k)},$$

and its partial derivative with respect to m is

$$(-1)^k \frac{x^{2k}}{2^{2k}.\,k!} D_m \frac{1}{(m+1)(m+2)\cdots(m+k)}.$$

$$\log \frac{1}{(m+1)(m+2)\dots(m+k)} = -[\log(m+1) + \log(m+2) + \cdots$$
$$+ \log(m+k)].$$

Take the D_m of both members and we have

$$D_m \frac{1}{(m+1)\,(m+2)\cdots(m+k)}$$
$$= -\frac{1}{(m+1)(m+2)\cdots(m+k)}\left[\frac{1}{m+1} + \frac{1}{m+2} + \cdots \frac{1}{m+k}\right].$$

$$D_m\left[1 - \frac{x^2}{2^2(m+1)} + \frac{x^4}{2^4.2!(m+1)(m+2)} - \frac{x^6}{2^6.3!(m+1)(m+2)(m+3)}\right.$$
$$\left.+ \cdots\right] = \frac{x^2}{2^2}\frac{1}{(m+1)^2} - \frac{x^4}{2^4.2!}\frac{1}{(m+1)(m+2)}\left[\frac{1}{m+1} + \frac{1}{m+2}\right]$$
$$+ \frac{x^6}{2^6.3!}\frac{1}{(m+1)(m+2)(m+3)}\left[\frac{1}{m+1} + \frac{1}{m+2} + \frac{1}{m+3}\right] + \cdots$$

and we have

$$[D_m F(m,x)]_{m=0} = J_0(x)\log x + \frac{x^2}{2^2(1!)^2}\frac{1}{1} - \frac{x^4}{2^4(2!)^2}\left(\frac{1}{1} + \frac{1}{2}\right) + \frac{x^6}{2^6(3!)^2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right)$$
$$- \frac{x^8}{2^8(4!)^2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \cdots;$$

and
$$z = AJ_0(x) + BK_0(x), \tag{9}$$

where
$$K_0(x) = J_0(x)\log x + \frac{x^2}{2^2} - \frac{x^4}{2^4(2!)^2}\left(\frac{1}{1} + \frac{1}{2}\right) + \frac{x^6}{2^6(3!)^2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right)$$
$$- \frac{x^8}{2^8(4!)^2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \cdots \tag{10}$$

is the general solution of Fourier's Equation (8).

$K_0(x)$ is known as a *Bessel's Function* of the *Second Kind*.

18. It is worth while to confirm the results of the last few articles by getting the general solutions of the equations in question by a different and familiar method.

The general solution of any ordinary linear differential equation of the second order can be obtained when a particular solution of the equation has been found [v. Int. Cal. p. 321, § 24 (a)].

The most general form of a homogeneous ordinary linear differential equation of the second order is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \tag{1}$$

where P and Q are functions of x. Suppose that

$$y = v \tag{2}$$

is a particular solution of (1). Substitute $y = vz$ in (1) and we get

$$v\frac{d^2z}{dx^2} + \left(2\frac{dv}{dx} + Pv\right)\frac{dz}{dx} = 0\,. \tag{3}$$

Call $\dfrac{dz}{dx} = z'$. Then (3) becomes

$$v\frac{dz'}{dx} + \left(2\frac{dv}{dx} + Pv\right)z' = 0\,, \tag{4}$$

a differential equation of the first order in which the variables can be separated. Multiply by dx and divide by vz' and (4) reduces to

$$\frac{dz'}{z'} + 2\frac{dv}{v} + Pdx = 0\,.$$

Integrate and we have

$$\log z' + \log v^2 + \int Pdx = C$$

or

$$z'v^2 = e^{C - \int Pdx} = Be^{-\int Pdx}\,,$$

$$z' = \frac{dz}{dx} = B\frac{e^{-\int Pdx}}{v^2}\,,$$

$$z = A + B\int\frac{e^{-\int Pdx}}{v^2}dx\,;$$

and

$$y = v\left(A + B\int\frac{e^{-\int Pdx}}{v^2}dx\right) \tag{5}$$

is the general solution of (1), the only arbitrary constants in the second member of (5) being those explicitly written, namely, A and B.

(a) Apply this formula to (1) Art. 14,

$$\frac{d^2z}{dx^2} + a^2z = 0\,; \tag{1}$$

given: $z = \cos ax$, as a particular solution. Substituting in (5) we have since $P = 0$

$$z = \cos ax \left(A + B \int \frac{dx}{\cos^2 ax} \right)$$

$$= \cos ax \left(A + \frac{B}{a} \tan ax \right)$$

$$= A \cos ax + B_1 \sin ax, \tag{2}$$

as the general solution of (1), and this agrees perfectly with (5) Art. 14.

(b) Take equation (1) Art. 15.

$$x^2 \frac{d^2 z}{dx^2} + 2x \frac{dz}{dx} - m(m+1)z = 0 ; \tag{1}$$

given: $z = x^m$, as a particular solution.

Here $P = \frac{2}{x}$, $\int P dx = 2 \log x = \log x^2$, and $e^{-\int P dx} = \frac{1}{x^2}$. Hence by (5)

$$z = x^m \left(A + B \int \frac{dx}{x^{2m+2}} \right) = x^m \left(A + \frac{B}{-2m-1} \frac{1}{x^{2m+1}} \right),$$

that is

$$z = Ax^m + \frac{B_1}{x^{m+1}} \tag{2}$$

is the general solution of (1), and agrees with (2) Art. 15.

(c) Take Legendre's Equation, (2) Art. 16.

$$(1 - x^2) \frac{d^2 z}{dx^2} - 2x \frac{dz}{dx} + m(m+1)z = 0 ; \tag{1}$$

given: $z = P_m(x)$, as a particular solution.

Here $P = \frac{-2x}{1-x^2}$, $\int P dx = \log(1 - x^2)$, and $e^{-\int P dx} = \frac{1}{1-x^2}$.

Hence by (5) $$z = P_m(x) \left(A + B \int \frac{dx}{(1-x^2)[P_m(x)]^2} \right) \tag{2}$$

is the general solution of (1) and must agree with (10) Art. 16, if m is an integer, and therefore

$$Q_m(x) = C P_m(x) \int \frac{dx}{(1-x^2)[P_m(x)]^2} \tag{3}$$

where C is as yet undetermined, and no constant term is to be understood with the integral in the second member.

(d) Take Bessel's Equation, (1) Art. 17.

$$\frac{d^2 z}{dx^2} + \frac{1}{x} \frac{dz}{dx} + \left(1 - \frac{m^2}{x^2} \right) z = 0 ; \tag{1}$$

given: $z = J_m(x)$, as a particular solution.

Here $P = \frac{1}{x}$, $\int P dx = \log x$, and $e^{-\int P dx} = \frac{1}{x}$. Hence by (5)

$$z = J_m(x)\left(A + B\int \frac{dx}{x[J_m(x)]^2}\right) \tag{2}$$

is the general solution of Bessel's Equation.

If $m = 0$ (2) becomes

$$z = J_0(x)\left(A + B\int \frac{dx}{x[J_0(x)]^2}\right) \tag{3}$$

and must agree with (9) Art. 17. Therefore

$$K_0(x) = CJ_0(x)\int \frac{dx}{x[J_0(x)]^2}, \tag{4}$$

where C is at present undetermined, and no constant term is to be taken with the integral.

The first considerable subject suggested by the problems which we have taken up in this introductory chapter is that of development in Trigonometric Series (v. Arts. 7 and 8).

CHAPTER II.

DEVELOPMENT IN TRIGONOMETRIC SERIES.

19. We have seen in Chapter I. that it is sometimes important to be able to express a given function of a variable x, in terms of the sines or of the cosines of multiples of x. The problem in its general form was first solved by Fourier in his "Analytic Theory of Heat" (1822), and its solution plays a very important part in most branches of modern Physics. Series involving only sines and cosines of whole multiples of x, that is series of the form

$$b_0 + b_1 \cos x + b_2 \cos 2x + \cdots + a_1 \sin x + a_2 \sin 2x + \cdots$$

are generally known as Fourier's series.

Let us endeavor to develop a given function of x in terms of $\sin x$, $\sin 2x$, $\sin 3x$, &c., in such a way that the function and the series shall be equal for all values of x between $x = 0$ and $x = \pi$.

To fix our ideas let us suppose that we have a curve,

$$y = f(x),$$

given, and that we wish to form the equation,

$$y = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots,$$

of a curve which shall coincide with so much of the given curve as lies between the points corresponding to $x = 0$ and $x = \pi$.

It is clear that in the equation

$$y = a_1 \sin x \tag{1}$$

a_1 may be determined so that the curve represented shall pass through any given point. For if we substitute in (1) the coördinates of the point in question we shall have an equation of the first degree in which a_1 is the only unknown quantity and which will therefore give us one and only one value for a_1.

In like manner the curve

$$y = a_1 \sin x + a_2 \sin 2x$$

may be made to pass through any two arbitrarily chosen points whose abscissas lie between 0 and π provided that the abscissas are not equal; and

$$y = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots + a_n \sin nx$$

may be made to pass through any n arbitrarily chosen points whose abscissas lie between 0 and π provided as before that their abscissas are all different.

If, then, the given function $f(x)$ is of such a character that for each value of x between $x = 0$ and $x = \pi$ it has one and only one value, and if between $x = 0$ and $x = \pi$ it is finite and continuous, or if discontinuous has only *finite discontinuities* (v. Int. Cal. Art. 83, p. 78), the coefficients in

$$y = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots + a_n \sin nx \tag{2}$$

can be determined so that the curve represented by (2) will pass through any n arbitrarily chosen points of the curve

$$y = f(x) \tag{3}$$

whose abscissas lie between 0 and π and are all different, and these coefficients will have but one set of values.

For the sake of simplicity suppose that the n points are so chosen that their projections on the axis of X are equidistant.

Call $\frac{\pi}{n+1} = \Delta x$; then the coördinates of the n points will be $[\Delta x, f(\Delta x)]$, $[2\Delta x, f(2\Delta x)]$, $[3\Delta x, f(3\Delta x)]$, $\cdots$ $[n\Delta x, f(n\Delta x)]$. Substitute them in (2) and we have

$$\left.\begin{aligned}
f(\Delta x) &= a_1 \sin \Delta x + a_2 \sin 2\Delta x + a_3 \sin 3\Delta x + \cdots + a_n \sin n\Delta x \\
f(2\Delta x) &= a_1 \sin 2\Delta x + a_2 \sin 4\Delta x + a_3 \sin 6\Delta x + \cdots + a_n \sin 2n\Delta x \\
f(3\Delta x) &= a_1 \sin 3\Delta x + a_2 \sin 6\Delta x + a_3 \sin 9\Delta x + \cdots + a_n \sin 3n\Delta x \\
&\vdots \\
f(n\Delta x) &= a_1 \sin n\Delta x + a_2 \sin 2n\Delta x + a_3 \sin 3n\Delta x + \cdots + a_n \sin n^2\Delta x,
\end{aligned}\right\} \tag{4}$$

n equations of the first degree to determine the n coefficients a_1, a_2, a_3, $\cdots$ a_n.

Not only can equations (4) be solved in theory, but they can be actually solved in any given case by a very simple and ingenious method due to Lagrange.

Let us take as an example the simple problem to determine the coefficients a_1, a_2, a_3, a_4, and a_5, so that

$$y = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + a_4 \sin 4x + a_5 \sin 5x \tag{5}$$

shall pass through the five points of the line

$$y = x$$

which have the abscissas $\frac{\pi}{6}$, $\frac{2\pi}{6}$, $\frac{3\pi}{6}$, $\frac{4\pi}{6}$, and $\frac{5\pi}{6}$, $\frac{\pi}{6}$ here being Δx.

We must now solve the equations

$$\left.\begin{aligned}
\frac{\pi}{6} &= a_1 \sin \frac{\pi}{6} + a_2 \sin \frac{2\pi}{6} + a_3 \sin \frac{3\pi}{6} + a_4 \sin \frac{4\pi}{6} + a_5 \sin \frac{5\pi}{6} \\
\frac{2\pi}{6} &= a_1 \sin \frac{2\pi}{6} + a_2 \sin \frac{4\pi}{6} + a_3 \sin \frac{6\pi}{6} + a_4 \sin \frac{8\pi}{6} + a_5 \sin \frac{10\pi}{6} \\
\frac{3\pi}{6} &= a_1 \sin \frac{3\pi}{6} + a_2 \sin \frac{6\pi}{6} + a_3 \sin \frac{9\pi}{6} + a_4 \sin \frac{12\pi}{6} + a_5 \sin \frac{15\pi}{6} \\
\frac{4\pi}{6} &= a_1 \sin \frac{4\pi}{6} + a_2 \sin \frac{8\pi}{6} + a_3 \sin \frac{12\pi}{6} + a_4 \sin \frac{16\pi}{6} + a_5 \sin \frac{20\pi}{6} \\
\frac{5\pi}{6} &= a_1 \sin \frac{5\pi}{6} + a_2 \sin \frac{10\pi}{6} + a_3 \sin \frac{15\pi}{6} + a_4 \sin \frac{20\pi}{6} + a_5 \sin \frac{25\pi}{6}.
\end{aligned}\right\} \tag{6}$$

Multiply the first equation by $2\sin\frac{\pi}{6}$, the second by $2\sin\frac{2\pi}{6}$, the third by $2\sin\frac{3\pi}{6}$, the fourth by $2\sin\frac{4\pi}{6}$, the fifth by $2\sin\frac{5\pi}{6}$ and add the equations.

The coefficient of a_2 is

$$2\sin\frac{\pi}{6}\sin\frac{2\pi}{6}+2\sin\frac{2\pi}{6}\sin\frac{4\pi}{6}+2\sin\frac{3\pi}{6}\sin\frac{6\pi}{6}+2\sin\frac{4\pi}{6}\sin\frac{8\pi}{6}$$
$$+2\sin\frac{5\pi}{6}\sin\frac{10\pi}{6};$$

but $$2\sin\frac{\pi}{6}\sin\frac{2\pi}{6}=\cos\frac{\pi}{6}-\cos\frac{3\pi}{6}, \text{ \&c.}$$

Hence the coefficient of a_2 becomes

$$\left.\begin{array}{l}\cos\frac{\pi}{6}+\cos\frac{2\pi}{6}+\cos\frac{3\pi}{6}+\cos\frac{4\pi}{6}+\cos\frac{5\pi}{6}\\ -\cos\frac{3\pi}{6}-\cos\frac{6\pi}{6}-\cos\frac{9\pi}{6}-\cos\frac{12\pi}{6}-\cos\frac{15\pi}{6}\end{array}\right\} \quad (7)$$

and this may be reduced by the aid of an important Trigonometric formula which we proceed to establish.

20. LEMMA.

$$\cos\theta+\cos 2\theta+\cos 3\theta+\cdots+\cos n\theta=-\frac{1}{2}+\frac{1}{2}\frac{\sin(2n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}. \quad (1)$$

For let $S=\cos\theta+\cos 2\theta+\cos 3\theta+\cdots+\cos n\theta$ and multiply by $2\cos\theta$.

$$\begin{aligned}2S\cos\theta &= 2\cos^2\theta+2\cos\theta\cos 2\theta+2\cos\theta\cos 3\theta+\cdots+2\cos\theta\cos n\theta\\ &=1+\cos\theta+\cos 2\theta+\cdots+\cos(n-1)\theta\\ &\quad+\cos 2\theta+\cos 3\theta+\cos 4\theta+\cdots+\cos(n+1)\theta\\ &=2S+1+\cos(n+1)\theta-\cos\theta-\cos n\theta.\end{aligned}$$ Hence

$$S=-\frac{1}{2}+\frac{\cos n\theta-\cos(n+1)\theta}{2(1-\cos\theta)}$$

or $$S=-\frac{1}{2}+\frac{1}{2}\frac{\sin(2n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}.$$

Q.E.D.

21. Applying (1) Art. 20 to (7) Art. 19 the coefficient of a_2 **reduces to**

$$\frac{\sin\frac{11\pi}{12}}{2\sin\frac{\pi}{12}} - \frac{\sin\frac{33\pi}{12}}{2\sin\frac{3\pi}{12}};$$

but
$$\frac{11\pi}{12} = \pi - \frac{\pi}{12}, \text{ and } \frac{33\pi}{12} = 3\pi - \frac{3\pi}{12};$$

therefore
$$\frac{\sin\left(\pi - \frac{\pi}{12}\right)}{2\sin\frac{\pi}{12}} - \frac{\sin\left(3\pi - \frac{3\pi}{12}\right)}{2\sin\frac{3\pi}{12}} = \frac{1}{2} - \frac{1}{2} = 0,$$

and a_2 vanishes.

In like manner it may be shown that the coefficients of a_3, a_4, and a_5 vanish.

The coefficient of a_1 is

$$2\sin^2\frac{\pi}{6} + 2\sin^2\frac{2\pi}{6} + 2\sin^2\frac{3\pi}{6} + 2\sin^2\frac{4\pi}{6} + 2\sin^2\frac{5\pi}{6}$$

$$= 1 + 1 + 1 + 1 + 1$$
$$- \cos\frac{2\pi}{6} - \cos\frac{4\pi}{6} - \cos\frac{6\pi}{6} - \cos\frac{8\pi}{6} - \cos\frac{10\pi}{6}$$

$$= 5 + \frac{1}{2} - \frac{\sin\frac{11\pi}{6}}{2\sin\frac{\pi}{6}} = 5\tfrac{1}{2} - \frac{\sin\left(2\pi - \frac{\pi}{6}\right)}{2\sin\frac{\pi}{6}} = 6.$$

The first member of the final equation is

$\frac{2\pi}{6}\sin\frac{\pi}{6} + 2\frac{2\pi}{6}\sin\frac{2\pi}{6} + 2\frac{3\pi}{6}\sin\frac{3\pi}{6} + 2\frac{4\pi}{6}\sin\frac{4\pi}{6} + 2\frac{5\pi}{6}\sin\frac{5\pi}{6}$. Hence

$$a_1 = \frac{2}{6}\sum_{k=1}^{k=5}\frac{k\pi}{6}\sin\frac{k\pi}{6} = \frac{\pi}{6}(2+\sqrt{3}) = 2 \quad \text{approximately.}$$

If we multiply the first equation of (6) Art. 19 by $2\sin\frac{2\pi}{6}$, the second by $2\sin\frac{4\pi}{6}$, the third by $2\sin\frac{6\pi}{6}$, the fourth by $2\sin\frac{8\pi}{6}$, the fifth by $2\sin\frac{10\pi}{6}$, add and reduce as before we shall find

$$a_2 = \frac{2}{6}\sum_{k=1}^{k=5}\frac{k\pi}{6}\sin\frac{2k\pi}{6} = -\frac{\pi}{6}\sqrt{3} = -0.9;$$

and in like manner we get

$$a_3 = \frac{2}{6}\sum_{k=1}^{k=5}\frac{k\pi}{6}\sin\frac{3k\pi}{6} = \frac{\pi}{6} = 0.5$$

$$a_4 = \frac{2}{6}\sum_{k=1}^{k=5}\frac{k\pi}{6}\sin\frac{4k\pi}{6} = -\frac{\pi\sqrt{3}}{18} = -0.3$$

$$a_5 = \frac{2}{6}\sum_{k=1}^{k=5}\frac{k\pi}{6}\sin\frac{5k\pi}{6} = \frac{\pi}{6}(2-\sqrt{3}) = 0.1\,.$$

Therefore

$$y = 2\sin x - 0.9\sin 2x + 0.5\sin 3x - 0.3\sin 4x + 0.1\sin 5x \qquad (1)$$

cuts the curve $y = x$ at the five points whose abscissas are $\frac{\pi}{6}$, $\frac{2\pi}{6}$, $\frac{3\pi}{6}$, $\frac{4\pi}{6}$, and $\frac{5\pi}{6}$.

22. The equations (4) Art. 19 can be solved by exactly the same device. To find any coefficient a_m multiply the first equation by $2\sin m\Delta x$, the second by $2\sin 2m\Delta x$, the third by $2\sin 3m\Delta x$, &c. and add.

The coefficient of any other a as a_k in the resulting equation will be

$$2\sin k\Delta x\sin m\Delta x + 2\sin 2k\Delta x\sin 2m\Delta x + 2\sin 3k\Delta x\sin 3m\Delta x + \cdots$$
$$+ 2\sin nk\Delta x\sin nm\Delta x$$
$$= \cos(m-k)\Delta x + \cos 2(m-k)\Delta x + \cos 3(m-k)\Delta x + \cdots + \cos n(m-k)\Delta x$$
$$- \cos(m+k)\Delta x - \cos 2(m+k)\Delta x - \cos 3(m+k)\Delta x - \cdots - \cos n(m+k)\Delta x$$

$$= \frac{\sin\frac{2n+1}{2}(m-k)\Delta x}{2\sin\frac{(m-k)\Delta x}{2}} - \frac{\sin\frac{2n+1}{2}(m+k)\Delta x}{2\sin\frac{(m+k)\Delta x}{2}}\,; \qquad \text{by (1) Art. 20.}$$

$$\frac{2n+1}{2} = n + 1 - \frac{1}{2} \quad \text{and} \quad (n+1)\Delta x = \pi\,.$$

Hence the coefficient of a_k may be written

$$\frac{\sin\left[(m-k)\pi - \frac{(m-k)\Delta x}{2}\right]}{2\sin\frac{(m-k)\Delta x}{2}} - \frac{\sin\left[(m+k)\pi - \frac{(m+k)\Delta x}{2}\right]}{2\sin\frac{(m+k)\Delta x}{2}}\,.$$

but this is equal to $\frac{1}{2} - \frac{1}{2}$ or $-\frac{1}{2} + \frac{1}{2}$ according as $m - k$ is odd or even and so is zero in either case.

coefficient of a_m will be

$$2\sin^2 m\Delta x + 2\sin^2 2m\Delta x + 2\sin^2 3m\Delta x + \cdots + 2\sin^2 nm\Delta x$$

$$\begin{aligned} = \quad & 1 \quad + \quad 1 \quad + \quad 1 \quad + \cdots + \quad 1 \\ & -\cos 2m\Delta x - \cos 4m\Delta x - \cos 6m\Delta x - \cdots - \cos 2nm\Delta x \end{aligned}$$

$$= n + \frac{1}{2} - \frac{\sin(2n+1)m\Delta x}{2\sin m\Delta x}, \text{ by (1) Art. 20.}$$

$$(2n+1)m\Delta x = 2m(n+1)\Delta x - m\Delta x = 2m\pi - m\Delta x,$$

re $$\frac{\sin(2n+1)m\Delta x}{2\sin m\Delta x} = \frac{\sin(2m\pi - m\Delta x)}{2\sin m\Delta x} = -\frac{1}{2},$$

coefficient of a_m is $n+1$.

first member of our final equation will be

$$2\sum_{k=1}^{k=n} f(k\Delta x)\sin km\Delta x.$$

e

$$a_m = \frac{2}{n+1}\sum_{k=1}^{k=n} f(k\Delta x)\sin km\Delta x, \tag{1}$$

curve

$$y = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx, \tag{2}$$

he coefficients are given by (1) will pass through the n points of the $y = f(x)$ whose abscissas are $\Delta x, 2\Delta x, 3\Delta x, \cdots n\Delta x$, Δx being $\frac{\pi}{n+1}$.

ould be noted that since the n equations (4) Art. 19 are all of the first there will exist only one set of values for the n quantities a_1, a_2, a_3, that can satisfy these equations. Consequently the solution which we tained is the only solution possible.

The result just obtained obviously holds good no matter how great a f n may be taken.

w we suppose n indefinitely increased the two curves (2) Art. 22 and) will come nearer and nearer to coinciding throughout the whole of ortions between $x = 0$ and $x = \pi$, and consequently the limiting at equation (2) Art. 22 approaches as n is indefinitely increased will nt a curve absolutely coinciding between the values of x in question $= f(x)$.

Let us see what limiting value a_m approaches as n is indefinitely increased.

$$a_m = \frac{2}{n+1}\sum_{k=1}^{k=n} f(k\Delta x)\sin km\Delta x \qquad (1) \text{ Art. 22.}$$

$$= \frac{2\Delta x}{\pi}\sum_{k=1}^{k=n} f(k\Delta x)\sin km\Delta x$$

$$= \frac{2}{\pi}\left[f(\Delta x)\sin m\Delta x.\Delta x + f(2\Delta x)\sin 2m\Delta x.\Delta x + \cdots + f(n\Delta x)\sin nm\Delta x.\Delta x\right]$$

$$= \frac{2}{\pi}\left[f(\Delta x)\sin m\Delta x.\Delta x + f(2\Delta x)\sin 2m\Delta x.\Delta x + \cdots + f(\pi - \Delta x)\sin m(\pi - \Delta x).\Delta x\right]$$

since $\Delta x = \dfrac{\pi}{n+1}$.

As n is increased indefinitely Δx approaches zero as a limit. Hence the limiting value of a_m as n increases indefinitely is

$$\frac{2}{\pi}\lim_{\Delta x \doteq 0}\left[f(\Delta x)\sin m\Delta x.\Delta x + f(2\Delta x)\sin 2m\Delta x.\Delta x + \cdots + f(\pi - \Delta x)\sin m(\pi - \Delta x).\Delta x\right] *$$

$$= \frac{2}{\pi}\int_0^\pi f(x)\sin mx.dx. \qquad \text{[v. Int. Cal. Arts. 80, 81.]}$$

Hence $\quad f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots, \qquad (2)$

where any coefficient a_m is given by the formula

$$a_m = \frac{2}{\pi}\int_0^\pi f(x)\sin mx.dx, \qquad (3)$$

is a true development of $f(x)$ for all values of x between $x = 0$ and $x = \pi$ *provided that the series* (2) *is convergent*, for it is in that case only that we can assume that the limiting value of the second member of (2) Art. 22 can be obtained by adding the limiting values of the several terms.

When $x = 0$ and when $x = \pi$ every term in the second member of (2) is zero, and the second member is zero and will not be equal to $f(x)$ unless $f(x)$ is itself zero when $x = 0$ and $x = \pi$; but even when $f(x)$ is not zero for $x = 0$ and $x = \pi$ the development given above holds good for any value of x between zero and π no matter how near it may be taken to either of these values.

24. Instead of actually performing the elimination in equations (4) Art. 19 and getting a formula for a_m in terms of n, and then letting n increase indefinitely, we might have saved labor by the following method.

* We shall use the sign $\doteq$ for *approaches*. $\Delta x \doteq 0$ is read Δx approaches zero.

Return to equations (4) Art. 19 and multiply the first by $\Delta x \sin m\Delta x$, the second by $\Delta x \sin 2m\Delta x$, and so on, that is multiply each equation by Δx times the coefficient of a_m in that equation, and then add the equations.

We get as the coefficient of a_k

$$\sin k\Delta x \sin m\Delta x.\,\Delta x + \sin 2k\Delta x \sin 2m\Delta x.\,\Delta x + \cdots + \sin nk\Delta x \sin nm\Delta x.\,\Delta x.$$

Let us find its limiting value as n is indefinitely increased. It may be written, since $(n+1)\,\Delta x = \pi$,

$$\underset{\Delta x \doteq 0}{\text{limit}} \left[\sin k\Delta x \sin m\Delta x.\,\Delta x + \sin 2k\Delta x \sin 2m\Delta x.\,\Delta x + \cdots + \sin k(\pi - \Delta x) \sin m(\pi - \Delta x).\Delta x \right]$$

$$= \int_0^\pi \sin kx \sin mx.dx\,;$$

but $$\int_0^\pi \sin kx \sin mx.dx = \tfrac{1}{2}\int_0^\pi [\cos (m-k)x - \cos (m+k)x]dx$$

$$= 0 \text{ if } m \text{ and } k \text{ are not equal.}$$

The coefficient of a_m is

$$\Delta x(\sin^2 m\Delta x + \sin^2 2m\Delta x + \sin^2 3m\Delta x + \cdots + \sin^2 nm\Delta x).$$

Its limiting value

$$\underset{\Delta x \doteq 0}{\text{limit}} \left[\sin^2 m\Delta x.\Delta x + \sin^2 2m\Delta x.\Delta x + \cdots + \sin^2 m(\pi - \Delta x)\Delta x \right]$$

$$= \int_0^\pi \sin^2 mx.dx = \frac{\pi}{2}.$$

The first member is

$$f(\Delta x) \sin m\Delta x.\Delta x + f(2\Delta x) \sin 2m\Delta x.\Delta x + \cdots + f(n\Delta x) \sin mn\Delta x.\Delta x$$

and its limiting value is

$$\int_0^\pi f(x) \sin mx.dx\,.$$

Hence the limiting form approached by the final equation as n is increased is

$$\int_0^\pi f(x) \sin mx.dx = \frac{\pi}{2} a_m.$$

Whence $$a_m = \frac{2}{\pi}\int_0^\pi f(x) \sin mx.dx$$ as before.

This method is practically the same as *multiplying the equation*

$$f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \qquad (1)$$

by $\sin mx.dx$ *and integrating both members from zero to* π.

It is exceedingly important to realize that the short method of determining any coefficient a_m of the series (1) which has just been described in the italicized paragraph, is essentially the same as that of obtaining a_m by actual elimination from the equations (4) Art. 19, and then supposing n to increase indefinitely, thus making the curves (3) Art. 19 and (2) Art. 19 absolutely coincide between the values of x which are taken as the limits of the definite integration.

25. We see, then, that any function of x which is single-valued, finite, and continuous between $x = 0$ and $x = \pi$, or if discontinuous has only finite discontinuities each of which is preceded and succeeded by continuous portions, can probably be developed into a series of the form

$$f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \tag{1}$$

where
$$a_m = \frac{2}{\pi}\int_0^\pi f(x) \sin mx.dx = \frac{2}{\pi}\int_0^\pi f(a) \sin ma.da\,; \tag{2}$$

and the series and the function will be identical for all values of x between $x = 0$ and $x = \pi$, not including the values $x = 0$ and $x = \pi$ unless the given function is equal to zero for those values.

An elaborate investigation of the question of the convergence of the series (1), for which we have not space, entirely confirms the result formulated above * and shows in addition that at a point of finite discontinuity the series has a value equal to half the sum of the two values which the function approaches as we approach the point in question from opposite sides.

The investigation which we have made in the preceding sections establishes the fact that the curve represented by $y = f(x)$ need not follow the same mathematical law throughout its length, but may be made up of portions of entirely different curves. For example, a broken line or a locus consisting of finite parts of several different and disconnected straight lines can be represented perfectly well by $y =$ a sine series.

26. Let us obtain a few sine developments.

(*a*) Let
$$f(x) = x\,. \tag{1}$$

We have
$$x = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \tag{2}$$

where
$$a_m = \frac{2}{\pi}\int_0^\pi x \sin mx.dx \tag{3}$$

* Provided the function has not an infinite number of maxima and minima in the neighborhood of a point. v. Arts. 37–38.

$$\int x \sin mx.dx = \frac{1}{m^2}(\sin mx - mx \cos mx),$$

$$\int_0^\pi x \sin mx.dx = -\frac{(-1)^m \pi}{m},$$

and
$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right) \qquad (4)$$

(*b*) Let
$$f(x) = 1. \qquad (1)$$

$$a_m = \frac{2}{\pi}\int_0^\pi \sin mx.dx; \qquad (2)$$

$$\int \sin mx.dx = -\frac{\cos mx}{m},$$

$$\int_0^\pi \sin mx.dx = \frac{1}{m}(1 - \cos m\pi) = \frac{1}{m}[1 - (-1)^m]$$

$$= 0 \text{ if } m \text{ is even}$$

$$= \frac{2}{m} \text{ if } m \text{ is odd.}$$

Hence
$$1 = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right). \qquad (3)$$

It is to be noticed that (3) gives at once a sine development for any constant c. It is,

$$c = \frac{4c}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right). \qquad (4)$$

If we substitute $x = \frac{\pi}{2}$ in (4)(*a*) or (3)(*b*) we get a familiar result, namely

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots, \qquad (5)$$

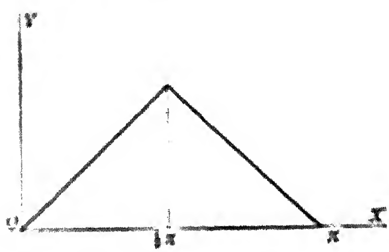
a formula usually derived by substituting $x = 1$ in the power series for $\tan^{-1}x$. (v. Dif. Cal. Art. 135.)

(4)(*a*) does not hold good when $x = \pi$, and (3)(*b*) fails when $x = 0$ and when $x = \pi$, for in all these cases the series reduces to zero.

(*c*) Let $f(x) = x$ from $x = 0$ to $x = \frac{\pi}{2}$

and $f(x) = \pi - x$ from $x = \frac{\pi}{2}$ to $x = \pi$.

That is, let $y = f(x)$ represent the broken line in the figure.



As the mathematical expression for $f(x)$ is different in the two halves of the curve we must break up

$$\int_0^{\pi} f(x)\sin mx.dx \quad \text{into} \quad \int_0^{\frac{\pi}{2}} f(x)\sin mx.dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\sin mx.dx.$$

We have, then,

$$a_m = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin mx.dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi - x)\sin mx.dx$$

$$= \frac{4}{m^2\pi}\sin m\frac{\pi}{2}.$$

But $\sin m\frac{\pi}{2} = 1$ if $m = 1$ or $4k+1$

$= 0$ " $m = 2$ " $4k+2$

$= -1$ " $m = 3$ " $4k+3$

$= 0$ " $m = 4$ " $4k$.

Hence if $y = f(x)$ represents our broken line,

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \cdots\right).$$

When $x = \frac{\pi}{2}$ $f(x) = \frac{\pi}{2}$ and we have

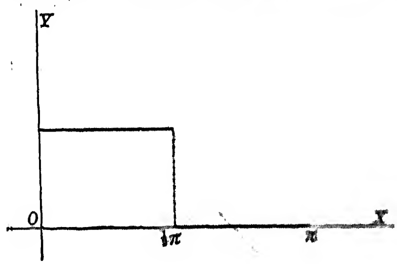
$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

(*d*) As a case where the function has a finite discontinuity, let

$f(x) = 1$ from $x = 0$ to $x = \frac{\pi}{2}$ and

$f(x) = 0$ " $x = \frac{\pi}{2}$ " $x = \pi$.

$y = f(x)$ will in this case represent the locus in the figure.



As before

$$\int_0^{\pi} f(x)\sin mx.dx = \int_0^{\frac{\pi}{2}} f(x)\sin mx.dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\sin mx.dx$$

$$a_m = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin mx.dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} 0.\sin mx.dx.$$

$$a_m = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin mx.dx = \frac{2}{\pi}\frac{1}{m}\left(1 - \cos m\frac{\pi}{2}\right).$$

But
$$\begin{aligned}
\cos m\frac{\pi}{2} &= 0 \quad \text{if} \quad m = 1 \quad \text{or} \quad 4k+1 \\
&= -1 \quad \text{"} \quad m = 2 \quad \text{"} \quad 4k+2 \\
&= 0 \quad \text{"} \quad m = 3 \quad \text{"} \quad 4k+3 \\
&= 1 \quad \text{"} \quad m = 4 \quad \text{"} \quad 4k.
\end{aligned}$$

Hence

$$f(x) = \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{2\sin 6x}{6} + \frac{\sin 7x}{7} + \cdots\right). \quad (2)$$

If $x = \frac{\pi}{2}$ the second member of (2) reduces to $\frac{1}{2}$, for

$$\frac{2}{\pi}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) = \frac{1}{2} \quad \text{by (5) } (b);$$

and we see that the series represents the function completely for all values of x between $x = 0$ and $x = \pi$ except for $x = \frac{\pi}{2}$ and there it has a value which is the mean of the values approached by the function as x approaches $\frac{\pi}{2}$ from opposite sides.

EXAMPLES.

Obtain the following developments: —

(1) $$x^2 = \frac{2}{\pi}\left[\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right)\sin x - \frac{\pi^2}{2}\sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin 3x - \frac{\pi^2}{4}\sin 4x + \left(\frac{\pi^2}{5} - \frac{4}{5^3}\right)\sin 5x - \cdots\right].$$

(2) $$x^3 = \frac{2}{\pi}\left[\left(\frac{\pi^3}{1} - \frac{6\pi}{1^3}\right)\sin x - \left(\frac{\pi^3}{2} - \frac{6\pi}{2^3}\right)\sin 2x + \left(\frac{\pi^3}{3} - \frac{6\pi}{3^3}\right)\sin 3x - \left(\frac{\pi^3}{4} - \frac{6\pi}{4^3}\right)\sin 4x + \cdots\right].$$

(3) $$f(x) = \frac{2}{\pi}\left[\frac{\sin x}{1^2} + \frac{\pi}{2}\sin 2x - \frac{\sin 3x}{3^2} - \frac{2\pi}{4^2}\sin 4x + \frac{\sin 5x}{5^2} + \frac{3\pi}{6^2}\sin 6x - \cdots\right],$$

if $f(x) = x$ from $x = 0$ to $x = \frac{\pi}{2}$ and $f(x) = 0$ from $x = \frac{\pi}{2}$ to $x = \pi$.

$$(4)\quad \sin\mu x = \frac{2}{\pi}\sin\mu\pi\left[\frac{\sin x}{1^2-\mu^2} - \frac{2\sin 2x}{2^2-\mu^2} + \frac{3\sin 3x}{3^2-\mu^2} - \frac{4\sin 4x}{4^2-\mu^2} + \cdots\right]$$

if μ is a fraction.

$$(5)\quad e^x = \frac{2}{\pi}\left[\frac{1}{2}(1+e^\pi)\sin x + \frac{2}{5}(1-e^\pi)\sin 2x + \frac{3}{10}(1+e^\pi)\sin 3x + \frac{4}{17}(1-e^\pi)\sin 4x + \cdots\right].$$

$$(6)\quad \sinh x = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2}\sin x - \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x - \frac{4}{17}\sin 4x + \cdots\right]$$

$$(7)\quad \cosh x = \frac{2}{\pi}\left[\frac{1}{2}(1+\cosh\pi)\sin x + \frac{2}{5}(1-\cosh\pi)\sin 2x + \frac{3}{10}(1+\cosh\pi)\sin 3x + \cdots\right].$$

27. Let us now try to develop a given function of x in a series of cosines. As before suppose that $f(x)$ has a single value for each value of x between $x = 0$ and $x = \pi$, that it does not become infinite between $x = 0$ and $x = \pi$, and that if discontinuous it has only finite discontinuities.

Assume

$$f(x) = b_0 + b_1\cos x + b_2\cos 2x + b_3\cos 3x + \cdots \quad (1)$$

To determine any coefficient b_m multiply (1) by $\cos mx.dx$ and integrate each term from 0 to π.

$$\int_0^\pi b_0\cos mx.dx = 0.$$

$$\int_0^\pi b_k\cos kx\cos mx.dx = \frac{b_k}{2}\int_0^\pi[\cos(m-k)x + \cos(m+k)x]dx$$

$$= 0 \text{ if } m \text{ and } k \text{ are not equal.}$$

$$\int b_m\cos^2 mx.dx = \frac{b_m}{2m}(mx + \cos mx\sin mx),$$

$$\int_0^\pi b_m\cos^2 mx.dx = \frac{\pi}{2}b_m, \quad \text{if } m \text{ is not zero.}$$

Hence $$b_m = \frac{2}{\pi}\int_0^\pi f(x)\cos mx.dx = \frac{2}{\pi}\int_0^\pi f(\alpha)\cos m\alpha.d\alpha,$$

if m is not zero.

To get b_0 multiply (1) by dx and integrate from zero to π.

$$\int_0^\pi b_0 dx = b_0\pi,$$

$$\int_0^\pi b_k \cos kx.dx = 0.$$

Hence
$$b_0 = \frac{1}{\pi}\int_0^\pi f(x)dx = \frac{1}{\pi}\int_0^\pi f(\alpha)d\alpha, \tag{3}$$

which is just half the value that would be given by formula (2) if zero were substituted for m.

To save a separate formula (1) is usually written

$$f(x) = \tfrac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots \tag{4}$$

and then the formula

$$b_m = \frac{2}{\pi}\int_0^\pi f(x)\cos mx.dx = \frac{2}{\pi}\int_0^\pi f(\alpha)\cos m\alpha.d\alpha \tag{2}$$

will give b_0 as well as the other coefficients.

It is important to see clearly that what we have just done in determining the coefficients of (1) is equivalent to taking $n+1$ terms of (4), substituting in

$$y = \tfrac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + \cdots + b_n \cos nx \tag{5}$$

in turn the coordinates of the $n+1$ points of the curve

$$y = f(x)$$

whose projections on the axis of X are equidistant, determining $b_0, b_1, b_2, \cdots b_n$ by elimination from the $n+1$ resulting equations, and then taking the limiting values they approach as n is indefinitely increased. (v. Art. 24.)

If $\Delta x = \dfrac{\pi}{n+1}$ the abscissas of the $n+1$ points used are $0, \Delta x, 2\Delta x, 3\Delta x, \cdots n\Delta x$, so that we should expect our cosine development to hold for $x = 0$ as well as for values of x between zero and π.

28. Let us take one or two examples:

(a) Let
$$f(x) = x. \tag{1}$$

$$b_0 = \frac{2}{\pi}\int_0^\pi x dx = \frac{2}{\pi}\frac{\pi^2}{2} = \pi.$$

$$b_m = \frac{2}{\pi}\int_0^\pi x \cos mx.dx = \frac{2}{m^2\pi}(\cos m\pi - 1) = \frac{2}{m^2\pi}[(-1)^m - 1].$$

Hence $$x=\frac{\pi}{2}-\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\frac{\cos 7x}{7^2}+\cdots\right). \qquad (2)$$

(2) holds good not only for values of x between zero and π but for $x=0$ and $x=\pi$ as well, since for these values we have

$$0=\frac{\pi}{2}-\frac{4}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right) \qquad (3)$$

and $$\pi=\frac{\pi}{2}+\frac{4}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right) \qquad (4)$$

which are true by Art. 26 (c)(3).

(b) Let $$f(x)=x\sin x. \qquad (1)$$

$$b_0=\frac{2}{\pi}\int_0^\pi x\sin x\,dx=\frac{2}{\pi}\pi=2,$$

$$b_1=\frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx=\frac{1}{\pi}\int_0^\pi x\sin 2x\,dx=-\frac{1}{2},$$

$$b_m=\frac{2}{\pi}\int_0^\pi x\sin x\cos mx\,dx=\frac{1}{\pi}\int_0^\pi[x\sin(m+1)x-x\sin(m-1)x]dx$$

$$=\frac{2}{(m-1)(m+1)} \quad \text{if } m \text{ is odd}$$

$$=-\frac{2}{(m-1)(m+1)} \quad \text{if } m \text{ is even}.$$

Hence

$$x\sin x=1-\frac{\cos x}{2}-\frac{2\cos 2x}{1.3}+\frac{2\cos 3x}{2.4}-\frac{2\cos 4x}{3.5}+\cdots \qquad (2)$$

If $x=\frac{\pi}{2}$ we have

$$\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\cdots \qquad (3)$$

EXAMPLES.

Obtain the following developments:

(1) $$f(x)=\frac{\pi}{4}-\frac{2}{\pi}\left[\frac{\cos 2x}{1^2}+\frac{\cos 6x}{3^2}+\frac{\cos 10x}{5^2}+\frac{\cos 14x}{7^2}+\cdots\right]$$

if $f(x)=x$ from $x=0$ to $x=\frac{\pi}{2}$ and $f(x)=\pi-x$ from $x=\frac{\pi}{2}$ to $x=\pi$.

(2) $$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \cdots\right],$$

if $f(x) = 1$ from $x = 0$ to $x = \frac{\pi}{2}$ and $f(x) = 0$ from $x = \frac{\pi}{2}$ to $x = \pi$.

(3) $$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right].$$

(4) $$x^3 = \frac{\pi^3}{4} - \frac{6}{\pi}\left[\left(\frac{\pi^2}{1^2} - \frac{4}{1^4}\right)\cos x - \frac{\pi^2}{2^2}\cos 2x + \left(\frac{\pi^2}{3^2} - \frac{4}{3^4}\right)\cos 3x \right.$$
$$\left. - \frac{\pi^2}{4^2}\cos 4x + \left(\frac{\pi^2}{5^2} - \frac{4}{5^4}\right)\cos 5x - \cdots\right].$$

(5) $$f(x) = \frac{\pi}{8} + \frac{2}{\pi}\left[\left(\frac{\pi}{2} - 1\right)\cos x - \frac{2}{2^2}\cos 2x - \frac{1}{3^2}\left(\frac{3\pi}{2} + 1\right)\cos 3x \right.$$
$$\left. + \frac{1}{5^2}\left(\frac{5\pi}{2} - 1\right)\cos 5x - \frac{2}{6^2}\cos 6x - \cdots\right],$$

if $f(x) = x$ from $x = 0$ to $x = \frac{\pi}{2}$ and $f(x) = 0$ from $x = \frac{\pi}{2}$ to $x = \pi$.

(6) $$e^x = \frac{2}{\pi}\left[\frac{1}{2}(e^\pi - 1) - \frac{1}{1+1^2}(e^\pi + 1)\cos x + \frac{1}{1+2^2}(e^\pi - 1)\cos 2x \right.$$
$$\left. - \frac{1}{1+3^2}(e^\pi + 1)\cos 3x + \cdots\right].$$

(7) $$\cosh x = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \frac{1}{2}\cos x + \frac{1}{5}\cos 2x - \frac{1}{10}\cos 3x \right.$$
$$\left. + \frac{1}{17}\cos 4x - \cdots\right].$$

(8) $$\sinh x = \frac{2}{\pi}\left[\frac{1}{2}(\cosh\pi - 1) - \frac{1}{2}(\cosh\pi + 1)\cos x \right.$$
$$\left. + \frac{1}{5}(\cosh\pi - 1)\cos 2x - \frac{1}{10}(\cosh\pi + 1)\cos 3x + \cdots\right].$$

(9) $$\cos\mu x = \frac{2\mu\sin\mu\pi}{\pi}\left[\frac{1}{2\mu^2} - \frac{\cos x}{\mu^2 - 1^2} + \frac{\cos 2x}{\mu^2 - 2^2} - \frac{\cos 3x}{\mu^2 - 3^2} \right.$$
$$\left. + \frac{\cos 4x}{\mu^2 - 4^2} - \cdots\right],$$

if μ is a fraction.

29. Although any function can be expressed both as a sine series and as a cosine series, and the function and either series will be equal for all values of x between zero and π, there is a decided difference in the two series for other values of x.

Both series are periodic functions of x having the period 2π. If then we let y equal the series in question and construct the portion of the correspond-

ing curve which lies between the values $x = -\pi$ and $x = \pi$ the whole curve will consist of repetitions of this portion.

Since $\sin mx = -\sin(-mx)$ the ordinate corresponding to any value of x between $-\pi$ and zero in the sine curve will be the negative of the ordinate corresponding to the same value of x with the positive sign. In other words the curve

$$y = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \tag{1}$$

is symmetrical with respect to the origin.

Since $\cos mx = \cos(-mx)$ the ordinate corresponding to any value of x between $-\pi$ and zero in the cosine curve will be the same as the ordinate belonging to the corresponding positive value of x. In other words the curve

$$y = \tfrac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots \tag{2}$$

is symmetrical with respect to the axis of Y.

If then $f(x) = -f(-x)$, that is if $f(x)$ is an *odd* function the sine series corresponding to it will be equal to it for all values of x between $-\pi$ and π, except perhaps for the value $x = 0$ for which the series will necessarily be zero.

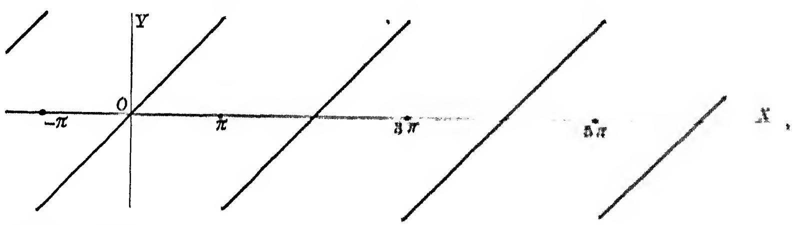
If $f(x) = f(-x)$, that is if $f(x)$ is an *even* function the cosine series corresponding to it will be equal to it for all values of x between $x = -\pi$ and $x = \pi$, not excepting the value $x = 0$.

As an example of the difference between the sine and cosine developments of the same function let us take the series for x.

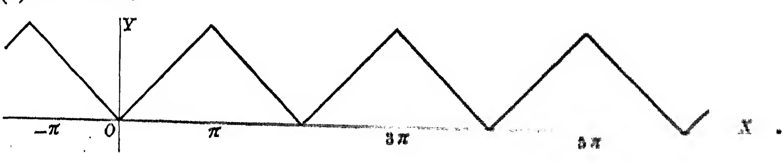
$$y = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right] \tag{3}$$

$$y = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \cdots\right] \tag{4}$$

[v. Art. 26(*a*) and Art. 28(*a*)]. (3) represents the curve



and (4) the curve



Both coincide with $y=x$ from $x=0$ to $x=\pi$, (3) coincides with $y=x$ from $x=-\pi$ to $x=\pi$, and neither coincides with $y=x$ for values of x less than $-\pi$ or greater than π. Moreover (3), in addition to the continuous portions of the locus represented in the figure, gives the isolated points $(-\pi,0)\,(\pi,0)\,(3\pi,0)$ &c.

30. We have seen that if $f(x)$ is an *odd* function its development in sine series holds for all values of x from $-\pi$ to π, as does the development of $f(x)$ in cosine series if $f(x)$ is an *even* function.

Thus the developments of Art. 26(a), Art. 26 Exs. (2), (4), (6); Art. 28(b) Art. 28 Exs. (3), (7), (9) are valid for all values of x between $-\pi$ and π.

Any function of x can be developed into a Trigonometric series to which it is equal for all values of x between $-\pi$ and π.

Let $f(x)$ be the given function of x. It can be expressed as the sum of an even function of x and an odd function of x by the following device.

$$f(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2} \tag{1}$$

identically; but $\dfrac{f(x)+f(-x)}{2}$ is not changed by reversing the sign of x and is therefore an *even* function of x; and when we reverse the sign of x, $\dfrac{f(x)-f(-x)}{2}$ is affected only to the extent of having its sign reversed and is consequently an *odd* function of x.

Therefore for all values of x between $-\pi$ and π

$$\frac{f(x)+f(-x)}{2}=\frac{1}{2}b_0+b_1\cos x+b_2\cos 2x+b_3\cos 3x+\cdots$$

where
$$b_m=\frac{2}{\pi}\int_0^\pi\frac{f(x)+f(-x)}{2}\cos mx.dx;$$
and

$$\frac{f(x)-f(-x)}{2}=a_1\sin x+a_2\sin 2x+a_3\sin 3x+\cdots$$

where
$$a_m=\frac{2}{\pi}\int_0^\pi\frac{f(x)-f(-x)}{2}\sin mx.dx.$$

b_m and a_m can be simplified a little

$$b_m=\frac{2}{\pi}\int_0^\pi\frac{f(x)+f(-x)}{2}\cos mx.dx$$

$$=\frac{1}{\pi}\left[\int_0^\pi f(x)\cos mx.dx+\int_0^\pi f(-x)\cos mx.dx\right].$$

but if we replace x by $-x$, we get

$$\int_0^\pi f(-x)\cos mx.dx = -\int_0^{-\pi} f(x)\cos mx.dx = \int_{-\pi}^0 f(x)\cos mx.dx$$

and we have $$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx.dx .$$

In the same way we can reduce the value of a_m to

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx.dx .$$

Hence

$$(2)\quad \begin{cases} f(x) = \frac{1}{2}b_0 + b_1\cos x + b_2\cos 2x + b_3\cos 3x + \dots \\ \qquad\qquad + a_1\sin x + a_2\sin 2x + a_3\sin 3x + \dots \end{cases}$$

where $$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx.dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cos m\alpha.d\alpha .$$

and $$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx.dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\sin m\alpha.d\alpha .$$

and this development holds for all values of x between $-\pi$ and π.

The second member of (2) is known as a Fourier's Series.

EXAMPLES.

1. Obtain the following developments, all of which are valid from $x = -\pi$ to $x = \pi$:—

(1) $$e^x = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \frac{1}{2}\cos x + \frac{1}{5}\cos 2x - \frac{1}{10}\cos 3x + \frac{1}{17}\cos 4x - \dots\right]$$
$$+ \frac{2\sinh\pi}{\pi}\left[\frac{1}{2}\sin x - \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x - \frac{4}{17}\sin 4x + \dots\right]$$

(2) $$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \dots\right]$$
$$+ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots,$$

where $f(x) = 0$ from $x = -\pi$ to $x = 0$ and $f(x) = x$ from $x = 0$ to

(3) $$f(x) = -\frac{3\pi}{16} + \frac{1}{\pi}\left[\frac{1}{1^2}\cos x + \frac{2}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \frac{2}{6^2}\cos 6x + \cdots\right]$$
$$+ \frac{1}{\pi}\left[\left(\frac{3\pi}{2} - 1\right)\sin x - \frac{3\pi}{4}\sin 2x + \left(\frac{3\pi}{6} + \frac{1}{3^2}\right)\sin 3x - \frac{3\pi}{8}\sin 4x + \left(\frac{3\pi}{10} - \frac{1}{5^2}\right)\sin 5x - \cdots\right],$$

where $f(x) = -x$ from $x = -\pi$ to $x = 0$, $f(x) = 0$ from $x = 0$ to $x = \frac{\pi}{2}$, and $f(x) = x - \frac{\pi}{2}$ from $x = \frac{\pi}{2}$ to $x = \pi$.

2. Show that formula (2) Art. 30 can be written

$$f(x) = \frac{1}{2}c_0\cos\beta_0 + c_1\cos(x - \beta_1) + c_2\cos(2x - \beta_2) + c_3\cos(3x - \beta_3) + \cdots$$

where $$c_m = (a_m^2 + b_m^2)^{\frac{1}{2}} \quad\text{and}\quad \beta_m = \tan^{-1}\frac{a_m}{b_m}.$$

3. Show that formula (2) Art. 30 can be written

$$f(x) = \frac{1}{2}c_0\sin\beta_0 + c_1\sin(x + \beta_1) + c_2\sin(2x + \beta_2) + c_3\sin(3x + \beta_3) + \cdots$$

where $$c_m = (a_m^2 + b_m^2)^{\frac{1}{2}} \quad\text{and}\quad \beta_m = \tan^{-1}\frac{b_m}{a_m}.$$

31. In developing a function of x into a Trigonometric series it is often inconvenient to be held within the narrow boundaries $x = -\pi$ and $x = \pi$. Let us see if we cannot widen them.

Let it be required to develop a function of x into a Trigonometric series which shall be equal to $f(x)$ for all values of x between $x = -c$ and $x = c$.

Introduce a new variable

$$z = \frac{\pi}{c}x,$$

which is equal to $-\pi$ when $x = -c$ and to π when $x = c$.

$f(x) = f\left(\frac{c}{\pi}z\right)$ can be developed in terms of z by Art. 30 (2), (3), and (4).

We have

$$f\left(\frac{c}{\pi}z\right) = \frac{1}{2}b_0 + b_1\cos z + b_2\cos 2z + b_3\cos 3z + \cdots + a_1\sin z + a_2\sin 2z + a_3\sin 3z + \cdots \quad (1)$$

where $$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{c}{\pi}z\right)\cos mz.dz. \quad (2)$$

and $$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{c}{\pi}z\right)\sin mz.dz\,. \tag{3}$$

and (1) holds good from $z = -\pi$ to $z = \pi$.

Replace z by its value in terms of x and (1) becomes

$$\left.\begin{aligned} f(x) = \frac{1}{2}b_0 + b_1\cos\frac{\pi x}{c} + b_2\cos\frac{2\pi x}{c} + b_3\cos\frac{3\pi x}{c} + \ldots \\ + a_1\sin\frac{\pi x}{c} + a_2\sin\frac{2\pi x}{c} + a_3\sin\frac{3\pi x}{c} + \ldots \end{aligned}\right\} \tag{4}$$

The coefficients in (4) are the same as in (1), and (4) holds good from $x = -c$ to $x = c$.

Formulas (2) and (3) can be put into more convenient shape.

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{c}{\pi}z\right)\cos mz.dz = \frac{1}{\pi}\int_{-c}^{c} f(x)\cos\frac{m\pi x}{c}\frac{\pi}{c}dx$$

or $$b_m = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{m\pi x}{c}dx = \frac{1}{c}\int_{-c}^{c} f(\lambda)\cos\frac{m\pi\lambda}{c}d\lambda\,. \tag{5}$$

In like manner we can transform (3) into

$$a_m = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{m\pi x}{c}dx = \frac{1}{c}\int_{-c}^{c} f(\lambda)\sin\frac{m\pi\lambda}{c}d\lambda\,. \tag{6}$$

By treating in like fashion formulas (1) and (2) Art. 25 and formulas (1) and (2) Art. 27 we get

$$f(x) = a_1\sin\frac{\pi x}{c} + a_2\sin\frac{2\pi x}{c} + a_3\sin\frac{3\pi x}{c} + \ldots \tag{7}$$

where $$a_m = \frac{2}{c}\int_0^c f(x)\sin\frac{m\pi x}{c}dx = \frac{2}{c}\int_0^c f(\lambda)\sin\frac{m\pi\lambda}{c}d\lambda\,. \tag{8}$$

and $$f(x) = \frac{1}{2}b_0 + b_1\cos\frac{\pi x}{c} + b_2\cos\frac{2\pi x}{c} + b_3\cos\frac{3\pi x}{c} + \ldots \tag{9}$$

where $$b_m = \frac{2}{c}\int_0^c f(x)\cos\frac{m\pi x}{c}dx = \frac{2}{c}\int_0^c f(\lambda)\cos\frac{m\pi\lambda}{c}d\lambda\,. \tag{10}$$

and (7) and (9) hold good from $x = 0$ to $x = c$.

EXAMPLES.

1. Obtain the following developments:

(1) $$1 = \frac{4}{\pi}\left[\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \cdots\right]$$

from $x = 0$ to $x = c$.

(2) $$x = \frac{2c}{\pi}\left[\sin\frac{\pi x}{c} - \frac{1}{2}\sin\frac{2\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} - \frac{1}{4}\sin\frac{4\pi x}{c} + \cdots\right]$$

from $x = -c$ to $x = c$.

$$x = \frac{c}{2} - \frac{4c}{\pi^2}\left[\cos\frac{\pi x}{c} + \frac{1}{3^2}\cos\frac{3\pi x}{c} + \frac{1}{5^2}\cos\frac{5\pi x}{c} + \frac{1}{7^2}\cos\frac{7\pi x}{c} + \cdots\right]$$

from $x = 0$ to $x = c$.

(3) $$x^2 = \frac{2c^2}{\pi^3}\left[\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right)\sin\frac{\pi x}{c} - \frac{\pi^2}{2}\sin\frac{2\pi x}{c} + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin\frac{3\pi x}{c}\right.$$
$$\left. - \frac{\pi^2}{4}\sin\frac{4\pi x}{c} + \left(\frac{\pi^2}{5} - \frac{4}{5^3}\right)\sin\frac{5\pi x}{c} + \cdots\right]$$

from $x = 0$ to $x = c$.

$$x^2 = \frac{c^2}{3} - \frac{4c^2}{\pi^2}\left[\cos\frac{\pi x}{c} - \frac{1}{2^2}\cos\frac{2\pi x}{c} + \frac{1}{3^2}\cos\frac{3\pi x}{c} - \frac{1}{4^2}\cos\frac{4\pi x}{c} + \cdots\right]$$

from $x = -c$ to $x = c$.

(4) $$e^x = 2\pi\left[\frac{1 + e^c}{c^2 + \pi^2}\sin\frac{\pi x}{c} + \frac{2(1 - e^c)}{c^2 + 4\pi^2}\sin\frac{2\pi x}{c}\right.$$
$$\left. + \frac{3(1 + e^c)}{c^2 + 9\pi^2}\sin\frac{3\pi x}{c} + \frac{4(1 - e^c)}{c^2 + 16\pi^2}\sin\frac{4\pi x}{c} + \cdots\right],$$

$$e^x = 2c\left[\frac{1}{2}\frac{e^c - 1}{c^2} - \frac{e^c + 1}{c^2 + \pi^2}\cos\frac{\pi x}{c} + \frac{e^c - 1}{c^2 + 4\pi^2}\cos\frac{2\pi x}{c}\right.$$
$$\left. - \frac{e^c + 1}{c^2 + 9\pi^2}\cos\frac{3\pi x}{c} + \cdots\right]$$

from $x = 0$ to $x = c$.

(5) $$f(x) = \frac{4c}{\pi^2}\left[\sin\frac{\pi x}{c} - \frac{1}{3^2}\sin\frac{3\pi x}{c} + \frac{1}{5^2}\sin\frac{5\pi x}{c} + \cdots\right]$$

from $x = 0$ to $x = c$,

where $f(x) = x$ from $x = 0$ to $x = \frac{c}{2}$ and $f(x) = c - x$ from $x = \frac{c}{2}$ to $x = c$.

2. Show that formula (4) Art. 31 can be written

$$f(x) = \frac{1}{2} c_0 \cos \beta_0 + c_1 \cos \left(\frac{\pi x}{c} - \beta_1 \right) + c_2 \cos \left(\frac{2\pi x}{c} - \beta_2 \right)$$
$$+ c_3 \cos \left(\frac{3\pi x}{c} - \beta_3 \right) + \cdots$$

where $$c_m = (a_m^2 + b_m^2)^{\frac{1}{2}} \quad \text{and} \quad \beta_m = \tan^{-1} \frac{a_m}{b_m}.$$

3. Show that formula (4) Art. 31 can be written

$$f(x) = \frac{1}{2} c_0 \sin \beta_0 + c_1 \sin \left(\frac{\pi x}{c} + \beta_1 \right) + c_2 \sin \left(\frac{2\pi x}{c} + \beta_2 \right)$$
$$+ c_3 \sin \left(\frac{3\pi x}{c} + \beta_3 \right) + \cdots$$

where $$c_m = (a_m^2 + b_m^2)^{\frac{1}{2}} \quad \text{and} \quad \beta_m = \tan^{-1} \frac{b_m}{a_m}.$$

32. In the formulas of Art. 31 c may have as great a value as we please, so that we can obtain a Trigonometric Series for $f(x)$ that will represent the given function through as great an interval as we may choose to take. If, then, we can obtain the limiting form approached by the series (4) Art. 31 as c is indefinitely increased the expression in question ought to be equal to the given function of x for all values of x. Equation (4) Art. 31 can be written as follows if we replace $b_0, b_1, b_2, \cdots a_1, a_2, \cdots$ by their values given in Art. 31 (5) and (6).

$$f(x) = \frac{1}{c} \left[\frac{1}{2} \int_{-c}^{c} f(\lambda)\, d\lambda \right.$$
$$+ \int_{-c}^{c} f(\lambda) \cos \frac{\pi\lambda}{c} \cos \frac{\pi x}{c}\, d\lambda + \int_{-c}^{c} f(\lambda) \cos \frac{2\pi\lambda}{c} \cos \frac{2\pi x}{c}\, d\lambda + \cdots$$
$$\left. + \int_{-c}^{c} f(\lambda) \sin \frac{\pi\lambda}{c} \sin \frac{\pi x}{c}\, d\lambda + \int_{-c}^{c} f(\lambda) \sin \frac{2\pi\lambda}{c} \sin \frac{2\pi x}{c}\, d\lambda + \cdots \right]$$
$$= \frac{1}{c} \int_{-c}^{c} f(\lambda)\, d\lambda \left[\frac{1}{2} + \cos \frac{\pi\lambda}{c} \cos \frac{\pi x}{c} + \sin \frac{\pi\lambda}{c} \sin \frac{\pi x}{c} \right.$$
$$\left. + \cos \frac{2\pi\lambda}{c} \cos \frac{2\pi x}{c} + \sin \frac{2\pi\lambda}{c} \sin \frac{2\pi x}{c} + \cdots \right]$$

$$f(x)=\frac{1}{c}\int_{-c}^{c} f(\lambda)d\lambda\left[\frac{1}{2}+\cos\frac{\pi}{c}(\lambda-x)+\cos\frac{2\pi}{c}(\lambda-x)+\cdots\right]$$

$$=\frac{1}{2c}\int_{-c}^{c} f(\lambda)d\lambda\left[1+\cos\frac{\pi}{c}(\lambda-x)+\cos\frac{2\pi}{c}(\lambda-x)+\cdots\right.$$

$$\left.+\cos\left(-\frac{\pi}{c}\right)(\lambda-x)+\cos\left(-\frac{2\pi}{c}\right)(\lambda-x)+\cdots\right]$$

since $\cos(-\phi)=\cos\phi$.

$$f(x)=\frac{1}{2\pi}\int_{-c}^{c} f(\lambda)d\lambda\left[\cdots+\frac{\pi}{c}\cos\left(-\frac{2\pi}{c}\right)(\lambda-x)+\frac{\pi}{c}\cos\left(-\frac{\pi}{c}\right)(\lambda-x)\right.$$

$$+\frac{\pi}{c}\cos\frac{0\pi}{c}(\lambda-x)+\frac{\pi}{c}\cos\frac{\pi}{c}(\lambda-x)$$

$$\left.+\frac{\pi}{c}\cos\frac{2\pi}{c}(\lambda-x)+\cdots\right] \tag{1}$$

As c is indefinitely increased the limiting value approached by the parenthesis in (1) is

$$\int_{-\infty}^{\infty}\cos\alpha(\lambda-x).d\alpha.$$

Hence the limiting form approached by (1) is

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\lambda)d\lambda\int_{-\infty}^{\infty}\cos\alpha(\lambda-x).d\alpha, \tag{2}$$

and the second member of (2) must be equal to $f(x)$ for all values of x.

The double integral in (2) is known as *Fourier's Integral*, and since it is a limiting form of *Fourier's Series* it is subject to the same limitations as the series.

That is, in order that (2) should be true $f(x)$ must be finite, continuous, and single valued for all values of x, or if discontinuous, must have only finite discontinuities.*

(2) is sometimes given in a slightly different form.

Since $\displaystyle\int_{-\infty}^{\infty}\cos\alpha(\lambda-x).d\alpha=\int_{-\infty}^{0}\cos\alpha(\lambda-x).d\alpha+\int_{0}^{\infty}\cos\alpha(\lambda-x).d\alpha$

and

$$\int_{-\infty}^{0}\cos\alpha(\lambda-x).d\alpha=\int_{\infty}^{0}\cos(-\alpha)(\lambda-x).d(-\alpha)=-\int_{\infty}^{0}\cos\alpha(\lambda-x).d\alpha$$

$$\int_{-\infty}^{\infty}\cos\alpha(\lambda-x).d\alpha=2\int_{0}^{\infty}\cos\alpha(\lambda-x).d\alpha$$

* See note on page 38.

and (2) may be written

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\lambda)d\lambda \int_0^{\infty} \cos a(\lambda - x).da\,. \tag{3}$$

If $f(x)$ is an *even* function or an *odd* function (3) can be still further simplified.

Let $$f(x) = -f(-x)\,.$$

Since the limits of integration in (3) do not contain a or λ the integrations may be performed in whichever order we choose. That is

$$\int_{-\infty}^{\infty} f(\lambda)d\lambda \int_0^{\infty} \cos a(\lambda - x).da = \int_0^{\infty} da \int_{-\infty}^{\infty} f(\lambda) \cos a(\lambda - x).d\lambda\,.$$

Now

$$\int_{-\infty}^{\infty} f(\lambda) \cos a(\lambda - x).d\lambda = \int_{-\infty}^{0} f(\lambda) \cos a(\lambda - x).d\lambda + \int_0^{\infty} f(\lambda) \cos a(\lambda - x).d\lambda\,.$$

$$\int_{-\infty}^{0} f(\lambda) \cos a(\lambda - x).d\lambda = \int_{\infty}^{0} f(-\lambda) \cos a(-\lambda - x).d(-\lambda)$$

$$= -\int_0^{\infty} f(\lambda) \cos a(\lambda + x).d\lambda$$

and (3) becomes

$$f(x) = \frac{1}{\pi}\int_0^{\infty} da \int_0^{\infty} f(\lambda)\,[\cos a(\lambda - x) - \cos a(\lambda + x)].d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} da \int_0^{\infty} f(\lambda) \sin a\lambda \sin ax.d\lambda$$

or

$$f(x) = \frac{2}{\pi}\int_0^{\infty} f(\lambda)d\lambda \int_0^{\infty} \sin a\lambda \sin ax.da\,. \tag{4}$$

If $f(x) = f(-x)$ (3) can be reduced in like manner to

$$f(x) = \frac{2}{\pi}\int_0^{\infty} f(\lambda)d\lambda \int_0^{\infty} \cos a\lambda \cos ax.da\,. \tag{5}$$

Although (4) holds for all values of x only in case $f(x)$ is an *odd* function, and (5) only in case $f(x)$ is an *even* function, both (4) and (5) hold for all *positive* values of x in the case of any function.

EXAMPLE.

(1) Obtain formulas (4) and (5) directly from (7) and (9) Art. 31.

CHAPTER III.

CONVERGENCE OF FOURIER'S SERIES.

33. The question of the *convergence* of a Fourier's Series is altogether too large to be completely handled in an elementary treatise. We will, however, consider at some length one of the most important of the series we have obtained, namely

$$\frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right]. \quad \text{[v. (3) Art. 26}(b).]$$

and prove that for all values of x between zero and π its sum is absolutely equal to unity; that is, that the limit approached by the sum of n terms of the series

$$\frac{2}{\pi}\left[\sin x\int_0^\pi \sin a.da + \sin 2x\int_0^\pi \sin 2a.da + \sin 3x\int_0^\pi \sin 3a.da + \cdots\right],$$

as n is indefinitely increased, is 1, provided that x lies between zero and π.

Let

$$S_n = \frac{2}{\pi}\left[\sin x\int_0^\pi \sin a.da + \sin 2x\int_0^\pi \sin 2a.da + \sin 3x\int_0^\pi \sin 3a.da + \cdots\right.$$

$$\left. + \sin nx\int_0^\pi \sin na.da\right]. \qquad (1)$$

Then

$$S_n = \frac{2}{\pi}\int_0^\pi [\sin a \sin x + \sin 2a \sin 2x + \sin 3a \sin 3x + \cdots + \sin na \sin nx]da$$

$$= \frac{1}{\pi}\int_0^\pi [\cos (a - x) - \cos (a + x) + \cos 2(a - x) - \cos 2(a + x) + \cdots$$

$$+ \cos n(a - x) - \cos n(a + x)]da$$

$$= \frac{1}{\pi}\int_0^\pi [\cos (a - x) + \cos 2(a - x) + \cos 3(a - x) + \cdots + \cos n(a - x)]da$$

$$- \frac{1}{\pi}\int_0^\pi [\cos (a + x) + \cos 2(a + x) + \cos 3(a + x) + \cdots + \cos n(a + x)]da.$$

Therefore by Art. 20 (1)

$$S_n = \frac{1}{\pi}\int_0^\pi \left[-\frac{1}{2} + \frac{1}{2}\frac{\sin(2n+1)\dfrac{a-x}{2}}{\sin\dfrac{a-x}{2}} \right] da$$

$$-\frac{1}{\pi}\int_0^\pi \left[-\frac{1}{2} + \frac{1}{2}\frac{\sin(2n+1)\dfrac{a+x}{2}}{\sin\dfrac{a+x}{2}} \right] da.$$

$$S_n = \frac{1}{2\pi}\int_0^\pi \frac{\sin(2n+1)\dfrac{a-x}{2}}{\sin\dfrac{a-x}{2}}\,da - \frac{1}{2\pi}\int_0^\pi \frac{\sin(2n+1)\dfrac{a+x}{2}}{\sin\dfrac{a+x}{2}}\,da.$$

In the first integral substitute β for $\dfrac{a-x}{2}$, and in the second integral substitute β for $\dfrac{a+x}{2}$.

We get

$$S_n = \frac{1}{\pi}\int_{-\frac{x}{2}}^{\frac{\pi}{2}-\frac{x}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta - \frac{1}{\pi}\int_{\frac{x}{2}}^{\frac{\pi}{2}+\frac{x}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta. \qquad (2)$$

It remains to find the limit approached by S_n as n is indefinitely increased.

34.

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta = \frac{\pi}{2}. \qquad (1)$$

For

$$\frac{\sin(2n+1)\beta}{2\sin\beta} = \tfrac{1}{2} + \cos 2\beta + \cos 4\beta + \cdots + \cos 2n\beta, \qquad \text{by Art. 20.}$$

and

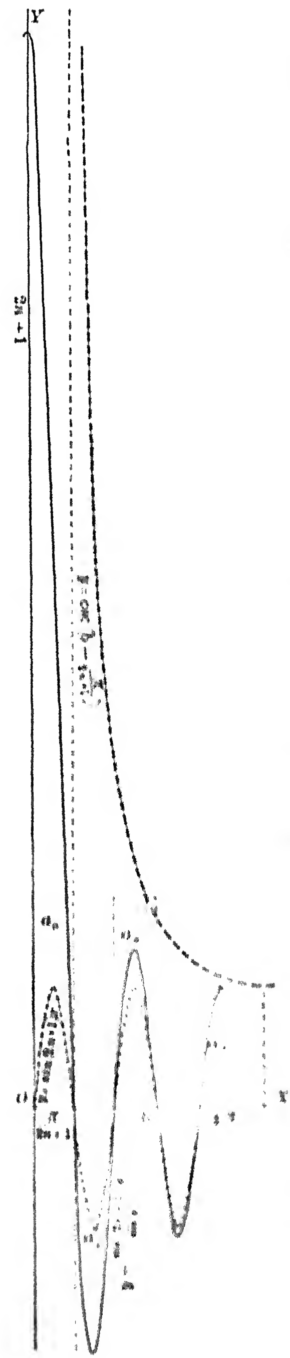
$$\int_0^{\frac{\pi}{2}} \cos 2k\beta . d\beta = 0.$$

Let us construct the curve

$$y = \frac{\sin(2n+1)x}{\sin x}.$$

We have only to draw the curve $y = \sin(2n+1)x$ and then to divide the length of each ordinate by the value of the sine of the corresponding abscissa.

In $y = \sin(2n+1)x$ the successive arches into which the curve is divided by the axis of X are equal, and consequently their areas are equal.



Each arch has for its altitude unity and for its base $\frac{\pi}{2n+1}$ and is symmetrical with respect to the ordinate of its highest or lowest point.

If now we form the curve $y = \frac{\sin(2n+1)x}{\sin x}$ from the curve $y = \sin(2n+1)x$, it is clear that, since $\sin x$ increases as x increases from 0 to $\frac{\pi}{2}$, the ordinate of any point of the new curve will be shorter than the ordinate of the corresponding point in the preceding arch, and that consequently the area of each arch of $y = \frac{\sin(2n+1)x}{\sin x}$ will be less than that of the arch before it.

If $a_0, a_1, a_2, \cdots a_{n-1}$ are the areas of the successive arches and a_n that of the incomplete arch terminated by the ordinate corresponding to $x = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}\,dx = a_0 - a_1 + a_2 - a_3 + \cdots.$$

But

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta = \frac{\pi}{2} \text{ by (1).}$$

Hence

$$\frac{\pi}{2} = a_0 - a_1 + a_2 - a_3 + a_4 - \cdots + a_n \quad \text{if } n \text{ is even,}$$

or

$$\frac{\pi}{2} = a_0 - a_1 + a_2 - a_3 + a_4 - \cdots - a_n \quad \text{if } n \text{ is odd.}$$

These equations can be written

$$\frac{\pi}{2} = a_0 + (-a_1 + a_2) + (-a_3 + a_4)$$
$$+ (-a_5 + a_6) + \cdots + (-a_{n-1} + a_n)$$

if n is even, and

$$\frac{\pi}{2} = a_0 + (-a_1 + a_2) + (-a_3 + a_4)$$
$$+ (-a_5 + a_6) + \cdots + (-a_{n-2} + a_{n-1}) + (-a_n)$$

if n is odd.

In either case each parenthesis is a negative quantity since

$$a_0 > a_1 > a_2 > a_3 \cdots > a_n,$$

and it follows that a_0 is greater than $\frac{\pi}{2}$.

Again

$$\frac{\pi}{2} = a_0 - a_1 + (a_2 - a_3) + (a_4 - a_5) + \cdots + (a_{n-2} - a_{n-1}) + a_n$$

if n is even and

$$\frac{\pi}{2} = a_0 - a_1 + (a_2 - a_3) + (a_4 - a_5) + \cdots + (a_{n-1} - a_n)$$

if n is odd.

In either case each parenthesis is positive and it follows that $a_0 - a_1$ is less than $\frac{\pi}{2}$.

Since

$$a_0 > \frac{\pi}{2} > a_0 - a_1,$$

a_0 and $a_0 - a_1$ differ from $\frac{\pi}{2}$ by less than they differ from each other, that is, by less than a_1.

In like manner we can show that $a_0 - a_1$ and $a_0 - a_1 + a_2$ differ from $\frac{\pi}{2}$ by less than a_2; and in general that $a_0 - a_1 + a_2 - a_3 + \cdots \pm a_k$ differs from $\frac{\pi}{2}$ by less than a_k; or even that

$$a_0 - a_1 + a_2 - a_3 + \cdots \pm \frac{a_k}{p}$$

differs from $\frac{\pi}{2}$ by less than a_k no matter what the value of p, provided p is greater than unity.

35. From what has been proved in the last article it follows that

$$\int_0^b \frac{\sin(2n+1)x}{\sin x}\, dx,$$

where b is some value between $\frac{\pi}{2n+1}$ and $\frac{\pi}{2}$, differs from $\frac{\pi}{2}$ by less than the area of the arch in which the ordinate of $y = \frac{\sin(2n+1)x}{\sin x}$ corresponding to $x = b$ falls if this ordinate divides an arch, or by less than the area of the arch next beyond the point $(b, 0)$ if the curve crosses the axis of X at that point.

The area of the arch in question is less than $\frac{\pi}{2n+1}$, its base, multiplied by $\frac{1}{\sin\left(b-\frac{\pi}{2n+1}\right)}$, a value greater than the length of its longest ordinate.

Therefore
$$\int_0^b \frac{\sin(2n+1)x}{\sin x}\,dx$$
differs from $\frac{\pi}{2}$ by less than $\frac{\pi}{2n+1}\,\frac{1}{\sin\left(b-\frac{\pi}{2n+1}\right)}$.

If now n is indefinitely increased $\frac{\pi}{2n+1}\,\frac{1}{\sin\left(b-\frac{\pi}{2n+1}\right)}$ approaches zero as its limit, and we get the very important result

$$\underset{n=\infty}{\text{limit}}\left[\int_0^b \frac{\sin(2n+1)x}{\sin x}\,dx\right]=\frac{\pi}{2} \qquad (1)$$

if $0<b<\frac{\pi}{2}$.

36. $$S_n=\frac{1}{\pi}\int_{-\frac{x}{2}}^{\frac{\pi}{2}-\frac{x}{2}}\frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta-\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\frac{x}{2}}\frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta. \qquad \text{[Art. 33. (2)]}$$

$$=\frac{1}{\pi}\int_{-\frac{x}{2}}^{0}\frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta+\frac{1}{\pi}\int_0^{\frac{\pi}{2}-\frac{x}{2}}\frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta$$

$$-\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta+\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta$$

$$-\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\frac{x}{2}}\frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta.$$

This last value for S_n can be somewhat simplified.
Substituting $\gamma=-\beta$ we get

$$\int_{-\frac{x}{2}}^{0}\frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta=-\int_{\frac{x}{2}}^{0}\frac{\sin(2n+1)\gamma}{\sin\gamma}\,d\gamma=\int_0^{\frac{x}{2}}\frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta.$$

Substituting $\gamma = \pi - \beta$ in

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\frac{x}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta$$

we h

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\frac{x}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta = -\int_{\frac{\pi}{2}}^{\frac{\pi}{2}-\frac{x}{2}} \frac{\sin(2n+1)\gamma}{\sin\gamma}\,d\gamma = \int_{\frac{\pi}{2}-\frac{x}{2}}^{\frac{\pi}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta - \int_0^{\frac{\pi}{2}-\frac{x}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta.$$

Hence

$$S_n = \frac{2}{\pi}\int_0^{\frac{x}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta - \frac{2}{\pi}\int_0^{\frac{\pi}{2}-\frac{x}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta = \frac{\pi}{2} \qquad \text{by (1) Art.}$$

$$\underset{n=\infty}{\text{limit}}\left[\int_0^{\frac{x}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta\right] = \frac{\pi}{2} \quad \text{if} \quad 0 < x < \pi \qquad \text{by (1) Art.}$$

and

$$\underset{n=\infty}{\text{limit}}\left[\int_0^{\frac{\pi}{2}-\frac{x}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta\right] = \frac{\pi}{2} \quad \text{if} \quad 0 < x < \pi \qquad \text{by (1) Art.}$$

Therefore $\underset{n=\infty}{\text{limit}}\,[S_n] = 1 + 1 - 1 = 1$ if $0 < x < \pi$

$$\frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right] = 1$$

for all values of x between zero and π.

37. By a somewhat long but not especially difficult extension of the soning just given it can be shown that if $f(x)$ is single-valued and between $x = -\pi$ and $x = \pi$, and has only a finite number of tinuities and of *maxima* and *minima* between $x = -\pi$ and $x = \pi$ *Fourier's Series*

$$\frac{1}{2}b_0 + b_1\cos x + b_2\cos 2x + b_3\cos 3x + \cdots$$
$$+ a_1\sin x + a_2\sin 2x + a_3\sin 3x + \cdots$$

where $$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\sin m\alpha . d\alpha$$

and $$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cos m\alpha . d\alpha ,$$

and that Fourier's Series only is equal to $f(x)$ for all values of x between $x = -\pi$ and $x = \pi$, *excepting the values of x corresponding to the discontinuities* of $f(x)$, and the values π and $-\pi$ if $f(\pi)$ is not equal to $f(-\pi)$; and that if c is a value of x corresponding to a discontinuity of $f(x)$, the value of the series when $x = c$ is

$$\frac{1}{2}\,\underset{\epsilon \doteq 0}{\text{limit}}\,[f(c-\epsilon) + f(c+\epsilon)];$$

and that if $f(\pi)$ is not equal to $f(-\pi)$ the value of the series when $x = -\pi$ and when $x = \pi$ is

$$\frac{1}{2}[f(-\pi) + f(\pi)].$$

If $f(x)$ while satisfying the conditions named in the preceding paragraph except for a finite number of values of x, becomes infinite for those values, the series is equal to the function except for the values of x in question provided that $\int_{-\pi}^{\pi} f(x)dx$ is finite and determinate. (v. Int. Cal. Arts. 83 and 84.)

38. The question of the convergency of a Fourier's Series and the conditions under which a function may be developed in such a series was first attacked successfully by Dirichlet in 1829, and his conclusions have been criticised and extended by later mathematicians, notably by Riemann, Heine, Lipschitz, and du Bois Reymond. It may be noted that the criticisms relate not to the sufficiency but to the necessity of Dirichlet's conditions.

An excellent résumé of the literature of the subject is given by Arnold Sachse in a short dissertation published by Gauthier Villars, Paris, 1880, entitled "Essai Historique sur la Représentation d'une Fonction Arbitraire d'une seule variable par une Série Trigonométrique."

39. A good deal of light is thrown on the peculiarities of trigonometric series by the attempt to construct approximately the curves corresponding to them.

If we construct $y = a_1 \sin x$ and $y = a_2 \sin 2x$ and add the ordinates of the points having the same abscissas we shall obtain points on the curve

$$y = a_1 \sin x + a_2 \sin 2x.$$

If now we construct $y = a_3 \sin 3x$ and add the ordinates to those of $y = a_1 \sin x + a_2 \sin 2x$ we shall get the curve

$$y = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x.$$

By continuing this process we get successive approximations to

$$y = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + a_4 \sin 4x + \cdots$$

Let us apply this method to a few of the series which we have obtained in Chapter II.

Take

$$y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots \tag{1}$$

$= 0$ when $x = 0$, $\dfrac{\pi}{4}$ from $x = 0$ to $x = \pi$, and 0 when $x = \pi$,

v. Art. 26 [b](3).

$$y = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right) \tag{2}$$

$= x$ from $x = 0$ to $x = \pi$, and 0 when $x = \pi$,

Art. 26[a](4).

$$y = \frac{4}{\pi}\left[\frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + \cdots\right] \tag{3}$$

$= x$ from $x = 0$ to $x = \dfrac{\pi}{2}$, and $\pi - x$ from $x = \dfrac{\pi}{2}$ to $x = \pi$,

Art. 26 [c](2).

$$y = \frac{1}{1}\sin x + \frac{2}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{2}{6}\sin 6x + \frac{1}{7}\sin 7x + \cdots \tag{4}$$

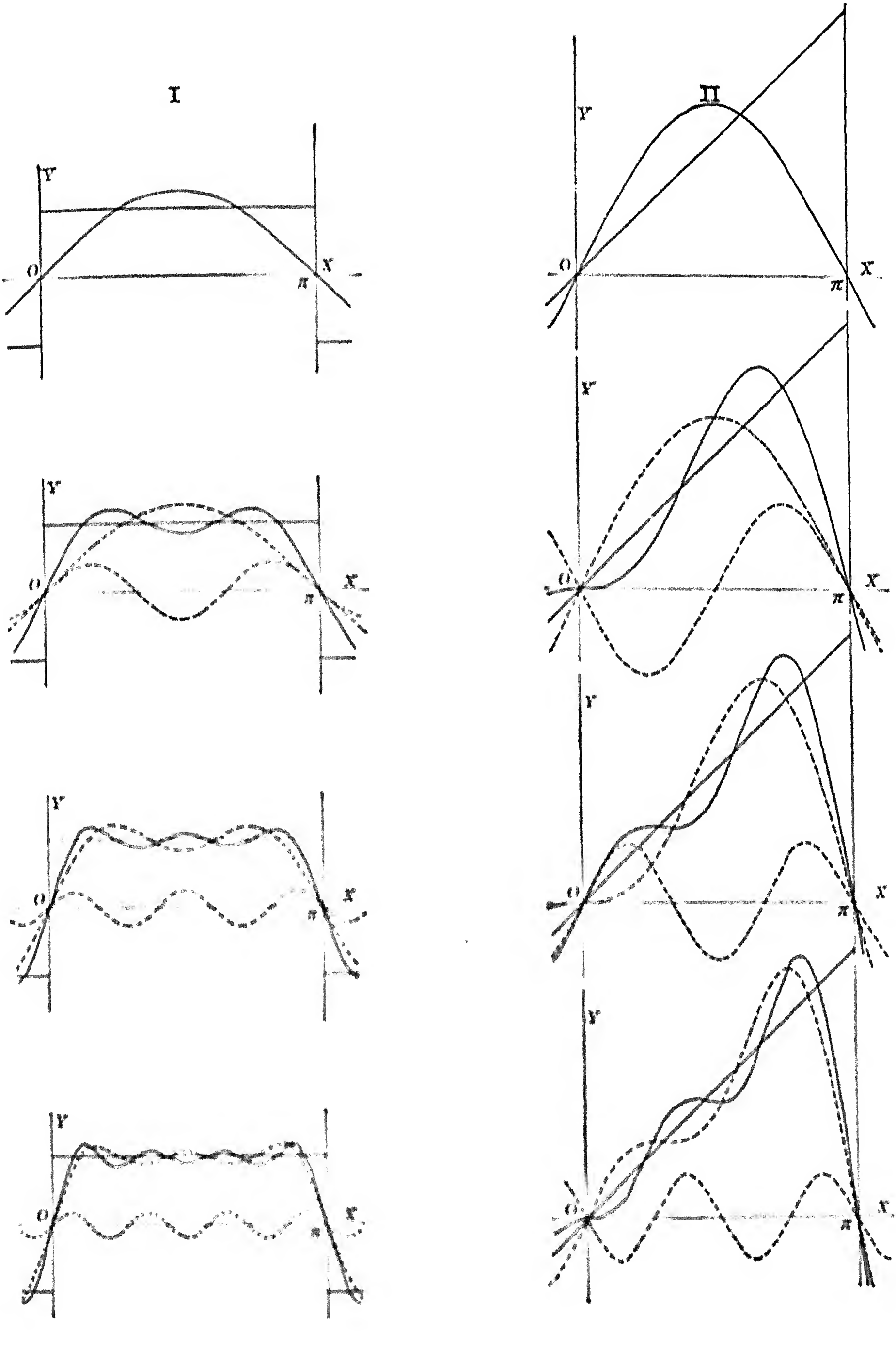
$= 0$ when $x = 0$, $\dfrac{\pi}{2}$ from $x = 0$ to $x = \dfrac{\pi}{2}$, and 0 from $x = \dfrac{\pi}{2}$ to $x = \pi$.

v. Art. 26 [d](2).

It must be borne in mind that each of these curves is periodic having the period 2π, and is symmetrical with respect to the origin.

The following figures I, II, III, and IV represent the first four approximations to each of these curves.

In each figure the curve $y =$ the series, and the approximation in question are drawn in continuous lines, and the preceding approximation and the curve corresponding to the term to be added are drawn in dotted lines.


I
II
Y
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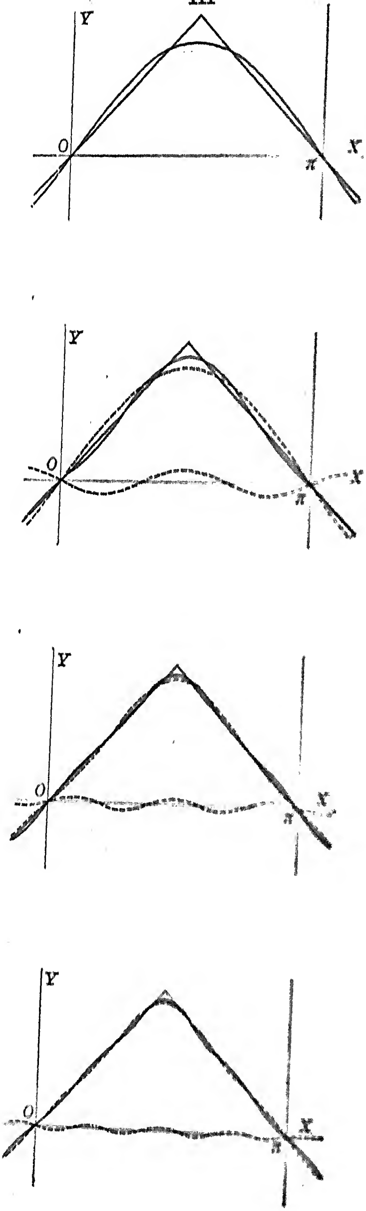
III



IV

Figs. I, II, III, and IV immediately suggest the following facts:

(*a*) The curve representing each approximation is continuous even when the curve representing the series is discontinuous.

(*b*) When the curve representing the series is discontinuous the portion of each successive approximate curve in the neighborhood of the point whose abscissa is a value of x for which the series curve is discontinuous approaches more and more nearly a straight line perpendicular to the axis of X and connecting the separate portions of the series curve.

(*c*) The curves representing successive approximations do not necessarily tend to lose their wavy character, since each is obtained from the preceding one by superposing upon it a wave line whose waves are shorter each time but do not necessarily lose their sharpness of pitch. This is the case in Figures I, II, and IV. In Fig. III the waves of the superposed curves grow rapidly flatter.

It follows from this that in such cases as those represented in Figures I, II, and IV the direction of the approximate curve at a point having a given abscissa does not in general approach the direction of the series curve at the corresponding point, or indeed, approach any limiting value, as the approximation is made closer and closer; and that the length of any portion of the approximate curve will not in general approach the length of the corresponding portion of the series curve.

Analytically this amounts to saying that the derivative of a function of x cannot in general be obtained by differentiating term by term the Fourier's Series which represents the function.

(*d*) The area bounded by a given ordinate, the approximate curve, the axis of X, and any second ordinate will approach as its limit the corresponding area of the series curve if the series curve is continuous between the ordinates in question; and will approach the area bounded by the given ordinate, the series curve, the axis of X, any second ordinate, and a line perpendicular to the axis of X, and joining the separate portions of the series curve if the latter has a discontinuity between the ordinates in question.

Analytically this amounts to saying that the Fourier's Series corresponding to any given function can be integrated term by term and the resulting series will represent the integral of the function even when the function is discontinuous (v. Int. Cal. Art. 83).

We may note in passing that if the function curve is continuous a curve representing the integral of the function will be continuous and will not change its direction abruptly at any point, while if the function curve is discontinuous the curve representing the integral will still be continuous but will change its direction abruptly at points corresponding to the discontinuities of the given function.

41. In general if we differentiate a Fourier's Series

$$S = \frac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots$$
$$+ a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots$$

we get

$$- b_1 \sin x - 2b_2 \sin 2x - 3b_3 \sin 3x - \cdots$$
$$+ a_1 \cos x + 2a_2 \cos 2x + 3a_3 \cos 3x + \cdots.$$

Differentiate again and we get

$$- b_1 \cos x - 2^2 b_2 \cos 2x - 3^2 b_3 \cos 3x - \cdots$$
$$- a_1 \sin x - 2^2 a_2 \sin 2x - 3^2 b_3 \sin 3x - \cdots.$$

We see that each time we differentiate we multiply the coefficient of $\sin kx$ and of $\cos kx$ by k while the term still involves $\cos kx$ or $\sin kx$.

Since the series

$$\cos x + \cos 2x + \cos 3x + \cdots$$
$$+ \sin x + \sin 2x + \sin 3x + \cdots$$

is not convergent, and a Fourier's Series converges only because its coefficients decrease as we advance in the series, the differentiation of a Fourier's Series must make its convergence less rapid if it does not actually destroy it, and repetitions of the process will usually eventually make the derived series diverge.

It is to be observed that the derived series are Fourier's Series, but of somewhat special form, that is they lack the constant term. (v. Art. 30.)

If now we integrate a Fourier's Series

$$\frac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots$$
$$+ a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots$$

we get

$$C + \frac{1}{2}b_0 x + b_1 \sin x + \frac{1}{2}b_2 \sin 2x + \frac{1}{3}b_3 \sin 3x + \cdots$$
$$- a_1 \cos x - \frac{1}{2}a_2 \cos 2x - \frac{1}{3}a_3 \cos 3x - \cdots,$$

a Trigonometric Series which converges more rapidly than the given series.

It is to be observed that the series obtained by integrating a Fourier's Series is not in general a Fourier's Series owing to the presence of the term $\frac{1}{2}b_0 x$. (v. Art. 30.)

42. We are now ready to consider the conditions under which a function of x can be developed into a Fourier's Series whose term by term derivative shall be equal to the derivative of the function.

Let the function $f(x)$ satisfy the conditions stated in Art. 37. Then there is one Fourier's Series and but one which is equal to it. Call this series S.

Let the derivative $f'(x)$* of the given function also satisfy the conditions stated in Art. 37. Then $f'(x)$ can be expressed as a Fourier's Series. By Art. 39 (*d*) the integral of this latter series will be equal to the integral of $f'(x)$, that is to $f(x)$ plus a constant, and one integral will be equal to $f(x)$.

If this integral which is necessarily a Trigonometric Series is a Fourier's Series it must be identical with S. It will be a Fourier's Series only in case the Fourier's Series for $f'(x)$ lacks the constant term $\frac{1}{2}b_0$.

But $$b_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)dx \qquad \text{by (3) Art. 30.}$$

Therefore $$b_0 = \frac{1}{\pi}\left[f(\pi) - f(-\pi)\right];$$

and will be zero if $f(\pi) = f(-\pi)$.

In order that $f'(x)$ shall satisfy the conditions stated in Art. 37 $f(x)$ while satisfying the same conditions must in addition be finite and continuous between $x = -\pi$ and $x = \pi$.

If, then, $f(x)$ is *single-valued, finite, and continuous, and has only a finite number of maxima and minima*, between $x = -\pi$ and $x = \pi$, (the values $x = -\pi$ and $x = \pi$ being included), *and if* $f(\pi) = f(-\pi)$ $f(x)$ can be developed into a Fourier's Series whose term by term derivative will be equal to the derivative of the function.

It will be observed that in this case the periodic curve $y = S$ is continuous throughout its whole extent.

43. Since a Fourier's Integral is a limiting case of a Fourier's Series the conclusions stated in this chapter hold, *mutatis mutandis* for a Fourier's Integral.

For example if a function of x is finite and single-valued for all values of x and has not an infinite number of discontinuities or of maxima and minima in the neighborhood of any value of x it will be equal to the Fourier's Integral

$$\frac{1}{\pi}\int_0^{\infty} da \int_{-\infty}^{\infty} f(\lambda)\cos a(\lambda - x).d\lambda$$

and to that Fourier's Integral only, and the integral with respect to x of this Fourier's Integral will be equal to $\int f(x)dx$.

If in addition $f(x)$ is finite and continuous for all values of x the derivative of the Fourier's Integral with respect to x will be equal to $\frac{df(x)}{dx}$.

* We shall regularly use the notation $f'(x)$ for $\frac{df(x)}{dx}$. v. Dif. Cal. Art. 124.

CHAPTER IV.

SOLUTION OF PROBLEMS IN PHYSICS BY THE AID OF FOURIER'S INTEGRALS AND FOURIER'S SERIES.

44. In Art. 7 we have already considered at some length a problem in Heat Conduction which required the use of a Fourier's Series. We shall begin the present chapter with a problem closely analogous in its treatment to that of Art. 7, but calling for the use of a Fourier's Integral.

Suppose that electricity is flowing in a thin plane sheet of infinite extent and that the value of the potential function is given for every point in some straight line in the sheet, required the value of the potential function at any point of the sheet.

Let us take the line as the axis of X and consider at first only those points for which y is positive.

We have, then, to satisfy the equation

$$D_x^2V + D_y^2V = 0 \tag{1}$$

subject to the conditions

$$V = 0 \quad \text{when} \quad y = \infty \tag{2}$$

$$V = f(x) \quad \text{"} \quad y = 0 \tag{3}$$

where $f(x)$ is a given function, and we are not concerned with negative values of y.

As in Art. 7 we have $e^{-\alpha y}\sin \alpha x$ and $e^{-\alpha y}\cos \alpha x$ as particular values of V which satisfy (1) and (2). We must multiply them by constant coefficients and so combine them as to satisfy condition (3).

By (3) Art. 32

$$f(x) = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^{\infty} f(\lambda)\cos \alpha(\lambda - x)\,d\lambda. \tag{4}$$

We wish to build up a value of V which will reduce to (4) when $y = 0$. This requires a little care but not much ingenuity.

Take $e^{-ay}\cos ax$ and $e^{-ay}\sin ax$ and multiply the first by $\cos a\lambda$, and the second by $\sin a\lambda$; they are still values of V which satisfy (1). Add these and we get

$$e^{-ay}\cos a(\lambda - x),$$

still a value of V which satisfies (1), no matter what the values of a and λ. Multiply by $f(\lambda)d\lambda$ and we have

$$e^{-ay}f(\lambda)\cos a(\lambda - x).d\lambda \tag{5}$$

as a value of V which satisfies (1).

$$V = \int_{-\infty}^{\infty} e^{-ay}f(\lambda)\cos a(\lambda - x).d\lambda \tag{6}$$

is still a solution of (1) since it is the limit of the sum of terms covered by the form (5); and finally

$$V = \frac{1}{\pi}\int_0^{\infty} da \int_{-\infty}^{\infty} e^{-ay}f(\lambda)\cos a(\lambda - x).d\lambda \tag{7}$$

is a solution of (1) as it is $\frac{1}{\pi}$ multiplied by the limit of the sum of terms formed by multiplying the second member of (6) by da and giving different values to a.

But (7) must be our required solution since while it satisfies (1) and (2), it reduces to (4) when $y = 0$ and therefore satisfies condition (3).

If $f(x)$ is an *even* function we can reduce (7) to the form

$$V = \frac{2}{\pi}\int_0^{\infty} da \int_0^{\infty} e^{-ay}f(\lambda)\cos ax\cos a\lambda.d\lambda \tag{8}$$

and if $f(x)$ is an *odd* function to the form

$$V = \frac{2}{\pi}\int_0^{\infty} da \int_0^{\infty} e^{-ay}f(\lambda)\sin ax\sin a\lambda.d\lambda. \tag{9}$$

(7), (8), and (9) are valid only for positive values of y, but as the problem is obviously symmetrical with respect to the axis of X, (7), (8), and (9) enable us to get the value of the potential function at any point of the plane.

EXAMPLES.

1. Obtain forms (8) and (9) directly by the aid of (5) and (4) Art. 32.

2. State a problem in statical electricity of which the solution given in Art. 44 is the solution.

45. As a special case under Art. 44 let us consider the problem: — To find the value of the potential function at any point of a thin plane sheet of infinite extent where all points of a given line which lie to the left of the origin are kept at potential zero, and all points which lie to the right of the origin are kept at potential unity.

Here $f(x) = 0$ if $x < 0$ and $f(x) = 1$ if $x > 0$.

(7) Art. 44 gives us the required solution. It is

$$V = \frac{1}{\pi}\int_0^\infty d\alpha \int_0^\infty e^{-\alpha y}\cos\alpha(\lambda - x).d\lambda; \tag{1}$$

but this can be much simplified.

We have

$$V = \frac{1}{\pi}\int_0^\infty d\lambda \int_0^\infty e^{-\alpha y}\cos\alpha(\lambda - x).d\alpha.$$

Now
$$\int_0^\infty e^{-ax}\cos mx.dx = \frac{a}{a^2 + m^2}$$
if $a > 0$. (Int. Cal. Art. 82, Ex. 8.)

Hence
$$\int_0^\infty e^{-\alpha y}\cos\alpha(\lambda - x).d\alpha = \frac{y}{y^2 + (\lambda - x)^2},$$

and
$$V = \frac{1}{\pi}\int_0^\infty \frac{y\,d\lambda}{y^2 + (\lambda - x)^2} = \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1}\frac{x}{y}\right).$$

$$\tan\left(\frac{\pi}{2} - \tan^{-1}\frac{x}{y}\right) = \text{ctn}\left(\tan^{-1}\frac{x}{y}\right) = \frac{y}{x};$$

and consequently

$$V = \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1}\frac{x}{y}\right) = 1 - \frac{1}{\pi}\tan^{-1}\frac{y}{x}. \tag{2}$$

Since
$$\log z = \log(x + yi) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{y}{x},$$

[Int. Cal. Art. 33 (2)],

$$i - \frac{1}{\pi}\log z = i - \frac{1}{\pi}\log(x + yi) = -\frac{1}{2\pi}\log(x^2 + y^2) + i\left(1 - \frac{1}{\pi}\tan^{-1}\frac{y}{x}\right)$$

and $1 - \frac{1}{\pi}\tan^{-1}\frac{y}{x}$ and $-\frac{1}{2\pi}\log(x^2 + y^2)$ are *conjugate functions*. (v. Int. Cal. Arts. 209 and 210.) Hence

$$V_1 = -\frac{1}{2\pi}\log(x^2 + y^2) \tag{3}$$

is a solution of the equation

$$D_x^2 V_1 + D_y^2 V_1 = 0; \tag{4}$$

and the curves

$$\frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1}\frac{x}{y}\right) = a \tag{5}$$

and

$$-\frac{1}{2\pi}\log(x^2 + y^2) = b \tag{6}$$

cut each other at right angles.

If we construct the curves obtained by giving different values to a in (5) we get a set of *equipotential lines* for the conducting sheet described at the beginning of this article, and the curves obtained by giving different values to (b) in (6) will be the *lines of flow*.

Moreover since

$$V_1 = -\frac{1}{2\pi}\log(x^2 + y^2) \tag{3}$$

is a solution of Laplace's Equation (1), the lines of flow just mentioned will be equipotential lines for a certain distribution of potential, for which the equipotential lines above mentioned will be lines of flow.

$V = a$, that is

$$\frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1}\frac{x}{y}\right) = a, \tag{5}$$

reduces to

$$y = -x\tan a\pi. \tag{7}$$

If now we give to a values differing by a constant amount we get a set of straight lines radiating from the origin and at equal angular intervals.

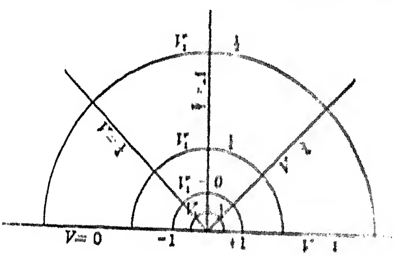
$V_1 = b$, that is

$$-\frac{1}{2\pi}\log(x^2 + y^2) = b, \tag{6}$$

reduces to

$$x^2 + y^2 = e^{-2\pi b}. \tag{8}$$

If we give to b a set of values differing by a constant amount we get a set of circles whose centres are at the origin and whose radii form a geometrical progression. They are the equipotential lines for a thin plane sheet of infinite extent where the potential function is kept equal to two different constant values on the circumferences of two given concentric circles, or where we have a source at the origin, and for this system the lines (7) are lines of flow, and (3) is the complete solution.



The figure gives the equipotential lines and lines of flow for either system, but only for positive values of y. The complete figure has the axis of X as an axis of symmetry.

EXAMPLES.

1. Solve the problem of Art. 44 for the case where

$$f(x) = -1 \quad \text{if} \quad x < 0 \quad \text{and} \quad f(x) = 1 \quad \text{if} \quad x > 0.$$

Ans., $\quad V = \dfrac{2}{\pi}\tan^{-1}\dfrac{x}{y}.$

2. Solve the problem of Art. 44 for the case where

$$f(x) = a \quad \text{if} \quad x < 0 \quad \text{and} \quad f(x) = b \quad \text{if} \quad x > 0.$$

Ans., $\quad V = \dfrac{1}{2}(a+b) + \dfrac{1}{\pi}(b-a)\tan^{-1}\dfrac{x}{y}.$

3. Reduce (7), (8), and (9) Art. 44 to the forms

$$V = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y f(\lambda)d\lambda}{y^2 + (\lambda - x)^2},$$

$$V = \frac{1}{\pi}\int_0^{\infty} y f(\lambda)d\lambda \left[\frac{1}{y^2 + (\lambda - x)^2} + \frac{1}{y^2 + (\lambda + x)^2}\right],$$

$$V = \frac{1}{\pi}\int_0^{\infty} y f(\lambda)d\lambda \left[\frac{1}{y^2 + (\lambda - x)^2} - \frac{1}{y^2 + (\lambda + x)^2}\right],$$

respectively.

46. An especially interesting case of Art. 44 is the following where $f(x) = 0$ if $x < -1$, $f(x) = 1$ if $-1 < x < 1$, and $f(x) = 0$ if $x > 1$.

Here $$V = \frac{1}{\pi}\left[\tan^{-1}\frac{1+x}{y} + \tan^{-1}\frac{1-x}{y}\right]. \tag{1}$$

Now $$\frac{1}{\pi}\log[(1-z)i] = \frac{1}{\pi}\log[(1-x-yi)i] = \frac{1}{\pi}\log[y+(1-x)i]$$

$$= \frac{1}{2\pi}\log[(1-x)^2+y^2] + \frac{i}{\pi}\tan^{-1}\frac{1-x}{y},$$

and

$$-\frac{1}{\pi}\log[(-1-z)i] = -\frac{1}{\pi}\log[(-1-x-yi)i] = -\frac{1}{\pi}\log[y-(1+x)i]$$

$$= -\frac{1}{2\pi}\log[(1+x)^2+y^2] + \frac{i}{\pi}\tan^{-1}\frac{1+x}{y}.$$

$$\frac{1}{\pi}\log\frac{1-z}{-1-z} = \frac{1}{2\pi}\log\frac{(1-x)^2+y^2}{(1+x)^2+y^2} + \frac{i}{\pi}\left[\tan^{-1}\frac{1+x}{y} + \tan^{-1}\frac{1-x}{y}\right].$$

Hence

$$\frac{1}{\pi}\left(\tan^{-1}\frac{1+x}{y}+\tan^{-1}\frac{1-x}{y}\right) \quad \text{and} \quad \frac{1}{2\pi}\log\frac{(1-x)^2+y^2}{(1+x)^2+y^2}$$

are *conjugate functions;** and

$$\frac{1}{\pi}\left(\tan^{-1}\frac{1+x}{y}+\tan^{-1}\frac{1-x}{y}\right)=a \tag{2}$$

is any equipotential line, and

$$\frac{1}{2\pi}\log\frac{(1-x)^2+y^2}{(1+x)^2+y^2}=b \tag{3}$$

any line of flow for the system described at the beginning of this article; and

$$V_1=\frac{1}{2\pi}\log\frac{(1-x)^2+y^2}{(1+x)^2+y^2} \tag{4}$$

is the solution of a new problem for which (3) represents any equipotential line and (2) any line of flow.

* The function conjugate to

$$\frac{1}{\pi}\left[\tan^{-1}\frac{1+x}{y}+\tan^{-1}\frac{1-x}{y}\right]$$

might have been found as follows. If ϕ is the required function and ψ the given function we have by Int. Cal. Arts. 211, 212, and 213 the relations

$$D_x\phi=D_y\psi \quad \text{and} \quad D_y\phi=-D_x\psi.$$

Here
$$D_y\psi=-\frac{1}{\pi}\left[\frac{1+x}{(1+x)^2+y^2}+\frac{1-x}{(1-x)^2+y^2}\right]$$

and
$$-D_x\psi=-\frac{1}{\pi}\left[\frac{y}{(1+x)^2+y^2}-\frac{y}{(1-x)^2+y^2}\right]$$

If now we integrate $D_y\psi$ with respect to x treating y as a constant and add an arbitrary function of y we shall have ϕ. So that

$$\phi=-\frac{1}{2\pi}\left\{\log[(1+x)^2+y^2]-\log[(1-x)^2+y^2]\right\}+f(y)$$

$$D_y\phi=-\frac{1}{\pi}\left[\frac{y}{(1+x)^2+y^2}-\frac{y}{(1-x)^2+y^2}\right]+\frac{df(y)}{dy}$$

Comparing this with its equal $-D_x\psi$ above we find $\frac{df(y)}{dy}=0$ and $f(y)=C$ a constant

therefore
$$\frac{1}{2\pi}\log\frac{(1-x)^2+y^2}{(1+x)^2+y^2}+C,$$

where C may be taken at pleasure, is our required conjugate function.

(2) reduces to

$$\frac{2y}{x^2+y^2-1} = \tan a\pi$$

or

$$x^2 + (y - \operatorname{ctn} a\pi)^2 = \csc^2 a\pi\,; \tag{5}$$

and (3) to

$$x^2 + y^2 + 2\,\frac{e^{2b\pi}+1}{e^{2b\pi}-1}\,x + 1 = 0$$

or

$$\left(x + \frac{e^{b\pi}+e^{-b\pi}}{e^{b\pi}-e^{-b\pi}}\right)^2 + y^2 = \left(\frac{e^{b\pi}+e^{-b\pi}}{e^{b\pi}-e^{-b\pi}}\right)^2 - 1$$

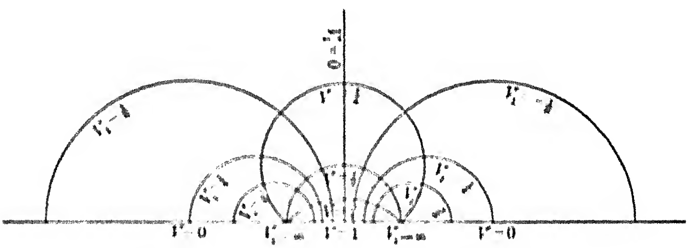
or

$$(x + \operatorname{ctnh} b\pi)^2 + y^2 = \operatorname{csch}^2 b\pi. \tag{6}$$

(5) and (6) are circles. The circles (5) have their centres in the axis of Y, and pass through the points $(-1, 0)$ and $(1, 0)$; and the circles (6) have their centres in the axis of X.

(4) is the complete solution, (6) is any equipotential line and (5) any line of flow for a plane sheet in which the points in the circumferences of two given circles whose centres are further apart than the sum of their radii are kept at different constant potentials, or where a source and a sink of equal intensity are placed at the points $(-1, 0)$ and $(1, 0)$. An important practical example is where two wires connected with the poles of a battery are placed with their free ends in contact with a thin plane sheet of conducting material. The figure shows the equipotential lines and lines of flow of either system.

The complete figure would have the axis of X for an axis of symmetry.



EXAMPLES.

1. Show that if $f(x) = a_1$ when $x < -b$, $f(x) = a_2$ when $-b < x < b$, $f(x) = a_3$ when $x > b$,

$$V = \frac{a_1 + a_3}{2} + \frac{1}{\pi}\left[(a_2 - a_1)\tan^{-1}\frac{b+x}{y} + (a_2 - a_3)\tan^{-1}\frac{b-x}{y}\right].$$

2. Show that if $f(x)=0$ if $x<0$, $f(x)=a_1$ if $0<x<b_1$, $f(x)=a_2$ if $b_1<x<b_2$, $f(x)=a_3$ if $b_2<x<b_3$, &c.,

$$V=\frac{1}{\pi}\Big[a_1\tan^{-1}\frac{x}{y}+(a_1-a_2)\tan^{-1}\frac{b_1-x}{y}+(a_2-a_3)\tan^{-1}\frac{b_2-x}{y}$$
$$+(a_3-a_4)\tan^{-1}\frac{b_3-x}{y}+\ldots\Big].$$

3. Show that if $f(x)=-1$ if $x<-1$, $f(x)=x$ if $-1<x<1$, $f(x)=1$ if $x>1$,

$$V=\frac{1}{\pi}\Big[(1+x)\tan^{-1}\frac{1+x}{y}-(1-x)\tan^{-1}\frac{1-x}{y}+\frac{y}{2}\log\frac{(1-x)^2+y^2}{(1+x)^2+y^2}\Big].$$

4. Show that if $f(x)=-1$ if $x<-1$, $f(x)=0$ if $-1<x<1$, $f(x)=1$ if $x>1$,

$$V=\frac{1}{\pi}\Big[\tan^{-1}\frac{1+x}{y}-\tan^{-1}\frac{1-x}{y}\Big].$$

Show that the equipotential lines are equilateral hyperbolas passing through the points $(-1, 0)$ and $(1, 0)$, and that the lines of flow are Cassinian ovals having $(-1, 0)$ and $(1, 0)$ as foci. The lines of flow are equipotential lines and the equipotential lines are lines of flow for the case where the points $(-1, 0)$ and $(1, 0)$ are kept at the same infinite potential, or where very small ovals surrounding these points are kept at the same finite potential. The case is approximately that of a pair of wires connected with the same pole of a battery whose other pole is grounded, and then placed with their ends in contact with a thin plane conducting sheet.

5. Show that if $f(x)=0$ if $x<0$, $f(x)=1$ if $0<x<a$, $f(x)=0$ if $a<x<b$, and $f(x)=1$ if $x>b$,

$$V=\frac{1}{\pi}\Big[\frac{\pi}{2}-\tan^{-1}\frac{a-x}{y}-\tan^{-1}\frac{b-x}{y}+\tan^{-1}\frac{x}{y}\Big].$$

The conjugate function

$$V=\frac{1}{2\pi}\log\frac{x^2+y^2}{[(a-x)^2+y^2][(b-x)^2+y^2]}$$

is the solution for the case where a sink and two sources of equal intensity lie on the axis of X, the sink at the origin and the sources at the distances a and b to the right of the origin. One of the lines of flow is easily seen to be the circle $x^2+y^2=ab$.

47. If the plane conducting sheet has two straight edges at right angles with each other and one is kept at potential zero while the value of the poten-

tial function is given at each point of the second, that is if $V=0$ when $x=0$ and $V=f(x)$ when $y=0$, the solution is readily obtained. It is

$$V=\frac{2}{\pi}\int_0^\infty da\int_0^\infty e^{-ay}f(\lambda)\sin ax\sin a\lambda.d\lambda. \tag{1}$$

v. (9) Art. 44.
This reduces to

$$V=\frac{1}{\pi}\int_0^\infty f(\lambda)d\lambda\left[\frac{y}{y^2+(\lambda-x)^2}-\frac{y}{y^2+(\lambda+x)^2}\right]. \tag{2}$$

v. Ex. 3 Art. 45.

EXAMPLES.

1. If $V=0$ when $y=0$ and $V=F(y)$ when $x=0$ show that

$$V=\frac{2}{\pi}\int_0^\infty da\int_0^\infty e^{-ax}F(\lambda)\sin ay\sin a\lambda.d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty F(\lambda)d\lambda\left[\frac{x}{x^2+(\lambda-y)^2}-\frac{x}{x^2+(\lambda+y)^2}\right].$$

2. If $V=f(x)$ when $y=0$ and $V=F(y)$ when $x=0$ show that

$$V=\frac{1}{\pi}\int_0^\infty\left[f(\lambda)\left(\frac{y}{y^2+(\lambda-x)^2}-\frac{y}{y^2+(\lambda+x)^2}\right)\right.$$

$$\left.+F(\lambda)\left(\frac{x}{x^2+(\lambda-y)^2}-\frac{x}{x^2+(\lambda+y)^2}\right)\right]d\lambda.$$

3. If $F(y)=b$ the result of Ex. 2 reduces to

$$V=\frac{2b}{\pi}\tan^{-1}\frac{y}{x}+\frac{1}{\pi}\int_0^\infty f(\lambda)d\lambda\left[\frac{y}{y^2+(\lambda-x)^2}-\frac{y}{y^2+(\lambda+x)^2}\right].$$

4. If $F(y)=1$ for $0<y<1$ and $F(y)=0$ for $y>1$ while $f(x)=1$ for $0<x<1$ and $f(x)=0$ for $x>1$

$$V=\frac{1}{\pi}\left[\tan^{-1}\frac{1-x}{y}-\tan^{-1}\frac{1+x}{y}+2\tan^{-1}\frac{y}{x}\right.$$

$$\left.+\tan^{-1}\frac{1-y}{x}-\tan^{-1}\frac{1+y}{x}+2\tan^{-1}\frac{x}{y}\right].$$

5. If one edge of the conducting sheet treated in Art. 47 is insulated, so that $D_xV=0$ if $x=0$ and $V=f(x)$ when $y=0$

$$V=\frac{2}{\pi}\int_0^\infty d\alpha\int_0^\infty e^{-\alpha y}f(\lambda)\cos\alpha x\cos\alpha\lambda . d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty f(\lambda)d\lambda\left[\frac{y}{y^2+(\lambda+x)^2}+\frac{y}{y^2+(\lambda-x)^2}\right].$$

48. If the conducting sheet is a long strip with parallel edges one of which is at potential zero while the value of the potential function is given at all points of the other, that is if $V=0$ when $y=0$ and $V=F(x)$ when $y=b$ the problem is not a very difficult one.

Since we are no longer concerned with the value of V when $y=\infty$ $V=e^{\alpha y}\sin\alpha x$ and $V=e^{\alpha y}\cos\alpha x$ are available as particular solutions of the equation

$$D_x^2V+D_y^2V=0 \qquad (1)$$

as well as $V=e^{-\alpha y}\sin\alpha x$ and $V=e^{-\alpha y}\cos\alpha x$.

Consequently $\quad \dfrac{e^{\alpha y}+e^{-\alpha y}}{2}\sin\alpha x=\cosh\alpha y\sin\alpha x \qquad$ [Int. Cal. Art. 43 (2)]

and $\quad \dfrac{e^{\alpha y}-e^{-\alpha y}}{2}\sin\alpha x=\sinh\alpha y\sin\alpha x \qquad$ [Int. Cal. Art. 43 (1)]

and $\quad \cosh\alpha y\cos\alpha x$ and $\sinh\alpha y\cos\alpha x$

are now available values of V and can be used precisely as $e^{-\alpha y}\cos\alpha x$ and $e^{-\alpha y}\sin\alpha x$ are used in Art. 44.

Following the same course as in Art. 44 we get

$$V=\frac{1}{\pi}\int_0^\infty d\alpha\int_{-\infty}^{\infty}\frac{\sinh\alpha y}{\sinh\alpha b}F(\lambda)\cos\alpha(\lambda-x)d\lambda \qquad (2)$$

as a solution of (1) which will reduce to $V=F(x)$ when $y=b$

and to $\quad V=0$ when $y=0$, since $\sinh 0=\dfrac{1-1}{2}=0$,

and (2) is therefore our required solution.

If V is to be equal to zero when $y=b$ and to $f(x)$ when $y=0$ we have only to replace y by $b-y$ and $F(x)$ by $f(x)$ in (2). We get

$$V=\frac{1}{\pi}\int_0^\infty d\alpha\int_{-\infty}^{\infty}\frac{\sinh\alpha(b-y)}{\sinh\alpha b}f(\lambda)\cos\alpha(\lambda-x)\,d\lambda. \qquad (3)$$

If $V=f(x)$ when $y=0$ and $V=F(x)$ when $y=b$ then

$$V=\frac{1}{\pi}\int_0^\infty da\int_{-\infty}^\infty \frac{\sinh a(b-y)}{\sinh ab} f(\lambda)\cos a(\lambda-x).d\lambda$$

$$+\frac{1}{\pi}\int_0^\infty da\int_{-\infty}^\infty \frac{\sinh ay}{\sinh ab} F(\lambda)\cos a(\lambda-x).d\lambda .$$

This can be considerably simplified by the aid of the formula

$$\int_0^\infty \frac{\sinh px}{\sinh qx}\cos rx.dx=\frac{\pi}{2q}\frac{\sin\frac{p\pi}{q}}{\cos\frac{p\pi}{q}+\cosh\frac{r\pi}{q}}$$

if $p^2<q^2$. [Bierens de Haan, Tables of Def. Int. (7) 265] and becomes

$$V=\frac{1}{2b}\sin\frac{\pi}{b}(b-y)\int_{-\infty}^\infty f(\lambda)\frac{d\lambda}{\cos\frac{\pi(b-y)}{b}+\cosh\frac{\pi}{b}(\lambda-x)}$$

$$+\frac{1}{2b}\sin\frac{\pi y}{b}\int_{-\infty}^\infty F(\lambda)\frac{d\lambda}{\cos\frac{\pi y}{b}+\cosh\frac{\pi}{b}(\lambda-x)}$$ or

$$V=\frac{1}{2b}\sin\frac{\pi y}{b}\int_{-\infty}^\infty\left[\frac{f(\lambda)}{\cosh\frac{\pi}{b}(\lambda-x)-\cos\frac{\pi y}{b}}+\frac{F(\lambda)}{\cosh\frac{\pi}{b}(\lambda-x)+\cos\frac{\pi y}{b}}\right]d\lambda. \quad (5)$$

EXAMPLES.

1. Given the formula

$$\int\frac{dx}{a+b\cosh x}=\frac{2}{\sqrt{b^2-a^2}}\tan^{-1}\left(\sqrt{\frac{b-a}{b+a}}\tanh\frac{x}{2}\right)\qquad \text{if } b>a,$$

show that if $V=1$ when $y=0$ and $V=0$ when $y=b$ $V=\frac{1}{b}(b-y)$.

2. Show that if $V=0$ when $y=b$, $V=-1$ when $y=0$ and $x<0$, and $V=1$ when $y=0$ and $x>0$

$$V=\frac{2}{\pi}\tan^{-1}\left[\frac{\tanh\frac{\pi x}{2b}}{\tan\frac{\pi y}{2b}}\right]$$

The solution for the conjugate system, that is, for a strip having a source at $(0,0)$ and an infinitely distant sink is

$$V=-\frac{1}{\pi}\log\left[\cosh^2\frac{\pi x}{2b}-\cos^2\frac{\pi y}{2b}\right].$$

3. Show that if $V=-1$ when $y=0$ and $x<0$, $V=1$ when $y=0$ and $x>0$, $V=-1$ when $y=b$ and $x<0$, and $V=1$ when $y=b$ and $x>0$,

$$V=\frac{2}{\pi}\tan^{-1}\left(\tan\frac{\pi}{2b}(b-y)\tanh\frac{\pi x}{2b}\right)+\frac{2}{\pi}\tan^{-1}\left(\tan\frac{\pi}{2b}y\tanh\frac{\pi x}{2b}\right)$$

$$=\frac{2}{\pi}\tan^{-1}\left[\frac{\sinh\frac{\pi x}{b}}{\sin\frac{\pi y}{b}}\right].$$

The solution for the conjugate system, that is, for a strip having a source and a sink at the points (0, 0) and (0, b) is

$$V=\frac{1}{\pi}\log\left[\frac{\cosh\frac{\pi x}{b}+\cos\frac{\pi y}{b}}{\cosh\frac{\pi x}{b}-\cos\frac{\pi y}{b}}\right].$$

4. If $V=0$ when $x=0$, $V=f(x)$ when $y=0$ and $x>0$, and $V=0$ when $y=b$ and $x>0$,

$$V=\frac{1}{\pi}\int_0^\infty da\int_0^\infty\frac{\sinh a(b-y)}{\sinh ab}\left[\cos a(\lambda-x)-\cos a(\lambda+x)\right].f(\lambda)d\lambda$$

$$=\frac{1}{2b}\sin\frac{\pi y}{b}\int_0^\infty\left[\frac{1}{\cosh\frac{\pi}{b}(\lambda-x)-\cos\frac{\pi y}{b}}-\frac{1}{\cosh\frac{\pi}{b}(\lambda+x)-\cos\frac{\pi y}{b}}\right].f(\lambda)d\lambda$$

for positive values of x and for values of y between 0 and b.

5. If $V_1=0$ when $x=0$, $V_1=F(x)$ when $y=b$ and $x>0$, and $V_1=0$ when $y=0$ and $x>0$

$$V_1=\frac{1}{2b}\sin\frac{\pi y}{b}\int_0^\infty\left[\frac{1}{\cosh\frac{\pi}{b}(\lambda-x)+\cos\frac{\pi y}{b}}-\frac{1}{\cosh\frac{\pi}{b}(\lambda+x)+\cos\frac{\pi y}{b}}\right]F(\lambda)d\lambda$$

for positive values of x and values of y between 0 and b.

6. If $V_2=0$ when $x=0$, $V_2=f(x)$ when $y=0$ and $x>0$, and $V_2=F(x)$ when $y=b$ and $x>0$

$$V_2=V+V_1 \quad \text{for} \quad x>0 \quad \text{and} \quad 0<y<b. \quad \text{(v. Exs. 4 and 5)}$$

7. If one edge of the strip described in Art. 48 is insulated so that we have $V=f(x)$ when $y=0$ and $D_yV=0$ when $y=b$ show that

$$V=\frac{1}{\pi}\int_0^\infty da\int_{-\infty}^\infty\frac{\cosh a(b-y)}{\cosh ab}f(\lambda)\cos a(\lambda-x).d\lambda.$$

By the aid of the formula

$$\int_0^\infty \frac{\cosh px}{\cosh qx}\cos rx . dx = \frac{\pi}{q}\,\frac{\cosh\dfrac{r\pi}{2q}\cos\dfrac{p\pi}{2q}}{\cos\dfrac{p\pi}{q}+\cosh\dfrac{r\pi}{q}} \qquad \text{if } p<q,$$

[Bierens de Haan, Def. Int. Tables (6) 265], reduce this to

$$V=\frac{1}{b}\sin\frac{\pi y}{2b}\int_{-\infty}^{\infty}\frac{f(\lambda)\cosh\dfrac{\pi}{2b}(\lambda-x)}{\cosh\dfrac{\pi}{b}(\lambda-x)-\cos\dfrac{\pi y}{b}}\,d\lambda.$$

8. If $V=0$ when $y=0$ or b and $x<-a$, $V=1$ when $y=0$ or b and $-a<x<a$, and $V=0$ when $y=0$ or b and $x>a$

$$V=\frac{1}{\pi}\left[\tan^{-1}\frac{\sinh\dfrac{\pi(a-x)}{b}}{\sin\dfrac{\pi y}{b}}+\tan^{-1}\frac{\sinh\dfrac{\pi(a+x)}{b}}{\sin\dfrac{\pi y}{b}}\right].$$

9. If $V=0$ when $y=0$ or b and $x<-a$, $V=1$ when $y=0$ and $-a<x<a$, $V=0$ when $y=0$ or b and $x>a$, and $V=-1$ when $y=b$ and $-a<x<a$

$$V=\frac{1}{\pi}\left[\tan^{-1}\frac{\tanh\dfrac{\pi(a-x)}{b}}{\tan\dfrac{\pi y}{b}}+\tan^{-1}\frac{\tanh\dfrac{\pi(a+x)}{b}}{\tan\dfrac{\pi y}{b}}\right].$$

10. A system conjugate to that of Ex. 9 is $V=+\infty$ when $y=0$ or b and $x=-a$, $V=-\infty$ when $y=0$ or b and $x=a$. In this case

$$V=\frac{1}{2\pi}\log\frac{\sin^2\dfrac{\pi y}{b}+\sinh^2\dfrac{\pi(a-x)}{b}}{\sin^2\dfrac{\pi y}{b}+\sinh^2\dfrac{\pi(a+x)}{b}}.$$

49. Let us take now a problem in the flow of heat. Suppose we have an infinite solid in which heat flows only in one direction, and that at the start the temperature of each point of the solid is given. Let it be required to find the temperature of any point of the solid at the end of the time t.

Here we have to solve the equation

$$D_t u = a^2 D_x^2 u \tag{1}$$

[v. Art. 1 (II)] subject to the condition

$$u = f(x) \quad \text{when} \quad t=0. \tag{2}$$

As the equation (1) is linear with constant coefficients we can get a particular solution by the device used in Arts. 7 and 8.

Let $u = e^{\beta t + \alpha x}$ and substitute in (1). We get

$$\beta = a^2\alpha^2$$

as the only relation which need hold between β and α.

Hence
$$u = e^{\alpha x + a^2\alpha^2 t} = e^{a^2\alpha^2 t}e^{\alpha x} \tag{3}$$
is a solution of (1) no matter what value is given to α.

To get a trigonometric form replace α by αi.

Then
$$u = e^{-a^2\alpha^2 t}e^{\alpha x i}.$$

If in (3) we replace α by $-\alpha i$ we get

$$u = e^{-a^2\alpha^2 t}e^{-\alpha x i}.$$

As in Arts. 7 and 8 we get from these values

$$u = e^{-a^2\alpha^2 t}\sin \alpha x \quad \text{and} \quad u = e^{-a^2\alpha^2 t}\cos \alpha x$$

as particular solutions of (1), α being wholly unrestricted.

From these values we wish to build up a value of u which shall reduce to $f(x)$ when $t = 0$ and shall still be a solution of (1).

We have
$$f(x) = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^{\infty} f(\lambda)\cos \alpha(\lambda - x).d\lambda \tag{4}$$
v. Art. 32 (3), and by proceeding as in Art. 44 we get

$$u = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^{\infty} e^{-a^2\alpha^2 t} f(\lambda)\cos \alpha(\lambda - x).d\lambda \tag{5}$$

as our required value of u.

This can be considerably simplified.

Changing the order of integration

$$u = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\lambda)d\lambda \int_0^\infty e^{-a^2\alpha^2 t}\cos \alpha(\lambda - x).d\alpha. \tag{6}$$

$$\int_0^\infty e^{-a^2\alpha^2 t}\cos \alpha(\lambda - x).d\alpha = \frac{1}{2a}\sqrt{\frac{\pi}{t}}.e^{-\frac{(\lambda - x)^2}{4a^2 t}} \tag{7}$$

by the formula

$$\int_0^\infty e^{-a^2x^2}\cos bx.dx = \frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{4a^2}} \qquad \text{[Int. Cal. Art. 94 (2)]}$$

Hence
$$u = \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty} f(\lambda)e^{-\frac{(\lambda - x)^2}{4a^2 t}}d\lambda. \tag{8}$$

Let now $$\beta = \frac{\lambda - x}{2a\sqrt{t}},$$

then $$\lambda = x + 2a\sqrt{t}.\beta$$

and $$u = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x + 2a\sqrt{t}.\beta)e^{-\beta^2}d\beta. \qquad (9)$$

EXAMPLES.

1. Let the solid be of infinite extent and let the temperature be equal to a constant c at the time $t = 0$.

Then $$u = \frac{c}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\beta^2}d\beta = \frac{2c}{\sqrt{\pi}}\int_0^{\infty} e^{-\beta^2}d\beta = c.$$

v. Int. Cal. Art. 92 (2).

2. Let $u = x$ when $t = 0$.

Then $$u = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(x + 2a\sqrt{t}.\beta)e^{-\beta^2}d\beta = x.$$

3. Let $u = x^2$ when $t = 0$.

Then $$u = x^2 + 2a^2t.$$

4. Let $u = 0$ if $x < -b$, $u = 1$ if $-b < x < b$, and $u = 0$ if $x > b$, when $t = 0$.

Then

$$u = \frac{1}{\sqrt{\pi}}\int_{-\frac{b+x}{2a\sqrt{t}}}^{\frac{b-x}{2a\sqrt{t}}} e^{-\beta^2}d\beta = \frac{2}{\sqrt{\pi}}\left[\frac{b}{2a\sqrt{t}} - \frac{b^3 + 3bx^2}{3\,(2a\sqrt{t})^3} + \frac{b^5 + 10b^3x^2 + 5bx^4}{5.2!(2a\sqrt{t})^5} - \cdots\right].$$

5. Let $u = 0$ if $x < 0$ and $u = 1$ if $x > 0$ when $t = 0$.

Then

$u = \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t}}} e^{-}$

$$u = \frac{1}{\sqrt{\pi}}\int_{-\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\beta^2}d\beta = \frac{1}{\sqrt{\pi}}\left[\int_0^{\frac{x}{2a\sqrt{t}}} e^{-\beta^2}d\beta + \int_0^{\infty} e^{-\beta^2}d\beta\right] = \frac{1}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t}}} e^{-\beta^2}d\beta + \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{\sqrt{\pi}}\left[\frac{x}{2a\sqrt{t}} - \frac{x^3}{3.(2a\sqrt{t})^3} + \frac{x^5}{5.2!(2a\sqrt{t})^5} - \frac{x^7}{7.3!(2a\sqrt{t})^7} + \cdots\right].$$

6. An iron slab 10 c. m. thick is placed between and in contact with two very thick iron slabs. The initial temperature of the middle slab is 100°, and of each of the outer slabs 0°. Required the temperature of a point in the middle of the inner slab fifteen minutes after the slabs have been put together. Given $a^2 = 0.185$ in C.G.S. units. *Ans.*, 21°.6.

7. Two very thick iron slabs one of which is at the temperature 0° and the other at the temperature 100° throughout are placed together face to face. Find the temperature of each slab 10 c. m. from their common face fifteen minutes after they have been placed together. *Ans.*, 70°.8, 29°.2.

8. Find a particular solution of $D_t u = a^2 D_x^2 u$ on the assumption that it is of the form $u = T.X$ where T is a function of t alone and X is a function of x alone.

50. If our solid has one plane face which is kept at the constant temperature zero, and we start with any given distribution of heat, the problem is somewhat modified.

Take the origin of coördinates in the plane face. Then we have as before the equation

$$D_t u = a^2 D_x^2 u, \tag{1}$$

but our conditions are

$$u = 0 \quad \text{when} \quad x = 0, \tag{2}$$

$$u = f(x) \quad " \quad t = 0, \tag{3}$$

and we are concerned only with positive values of x.

We may then use the form (4) Art. 32,

$$f(x) = \frac{2}{\pi}\int_0^\infty d\alpha \int_0^\infty f(\lambda) \sin \alpha x \sin \alpha\lambda \, d\lambda, \tag{4}$$

and proceeding as in the last section we get

$$u = \frac{2}{\pi}\int_0^\infty d\alpha \int_0^\infty e^{-a^2\alpha^2 t} f(\lambda) \sin \alpha x \sin \alpha\lambda \, d\lambda \tag{5}$$

as our required solution. This may be reduced considerably.

$$u = \frac{1}{\pi}\int_0^\infty f(\lambda)d\lambda \int_0^\infty e^{-a^2\alpha^2 t}[\cos \alpha(\lambda - x) - \cos \alpha(\lambda + x)]d\alpha,$$

or

$$u = \frac{1}{2a\sqrt{\pi t}}\int_0^\infty f(\lambda)\left[e^{-\frac{(\lambda - x)^2}{4a^2t}} - e^{-\frac{(\lambda + x)^2}{4a^2t}}\right]d\lambda \tag{6}$$

by (7) Art. 49, and this may be reduced to the form

$$u = \frac{1}{\sqrt{\pi}}\left[\int_{-\frac{x}{2a\sqrt{t}}}^\infty e^{-\beta^2} f(x + 2a\beta\sqrt{t})d\beta - \int_{\frac{x}{2a\sqrt{t}}}^\infty e^{-\beta^2} f(-x + 2a\beta\sqrt{t})d\beta\right]. \tag{7}$$

EXAMPLES.

1. Let the initial temperature be constant and equal to c.

Then

$$u = \frac{c}{\sqrt{\pi}}\left[\int_{-\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\beta^2}d\beta - \int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\beta^2}d\beta\right]$$

$$= \frac{2c}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t}}} e^{-\beta^2}d\beta$$

$$= \frac{2c}{\sqrt{\pi}}\left[\frac{x}{2a\sqrt{t}} - \frac{x^3}{3.(2a\sqrt{t})^3} + \frac{x^5}{5.2!(2a\sqrt{t})^5} - \frac{x^7}{7.3!(2a\sqrt{t})^7} + \cdots\right].$$

2. Assuming that the earth was originally at the temperature 7000° Fahrenheit throughout, and that the surface was kept at the constant temperature 0°, find (1) the temperature 10 miles below the surface 10,000,000 years after the cooling began; (2) the temperature 1 mile below the surface at the same epoch; (3) the temperature 10 miles below the surface 100,000,000 years after the cooling began; (4) the temperature 1 mile below the surface at the same epoch; (5) the rate at which the temperature was increasing with the distance from the surface at each point at each epoch.

Neglect the convexity of the earth's surface and take Sir Wm. Thomson's value of a^2 (400) the foot, the Fahrenheit degree, and the year being taken as units. (Thomson and Tait's Nat. Phil. Vol. II. Appendix.)

Ans., (1) 3114°; (2) 329°.5; (3) 1036°; (4) 103°; (5) 1° for every 20 feet, 3° for every 50 feet, 1° for every 50 feet, 1° for every 50 feet.

3. Let the initial temperature be constant and equal to $-b$, then by Ex. 1

$$u = -\frac{2b}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t}}} e^{-\beta^2}d\beta.$$

4. Let the temperature of the plane face be b instead of zero, and let the initial temperature be zero.

Then we have only to add b to the second member of the solution in Ex. 3, as we may since $u = b$ is a solution of (1) Art. 49, and we get

$$u = b\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t}}} e^{-\beta^2}d\beta\right).$$

5. Let $u = b$ when $x = 0$ and $u = f(x)$ when $t = 0$.

Then

$$u = b\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t}}} e^{-\beta^2}d\beta\right) + \frac{1}{2a\sqrt{\pi t}}\int_0^{\infty} f(\lambda)\left[e^{-\frac{(\lambda-x)^2}{4a^2t}} - e^{-\frac{(\lambda+x)^2}{4a^2t}}\right]d\lambda$$

by (6) Art. 50.

6. Let $u=b$ when $x=0$ and $u=c$ when $t=0$.

Then
$$u=b+(c-b)\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t}}}e^{-\beta^2}d\beta.$$

7. If the earth has been cooling for 200,000,000 years from a uniform temperature, prove that the rate of cooling is greatest at a depth of about 76 miles, and that at a depth of about 130 miles the rate of cooling has reached its maximum value for all time. Let $a^2=400$.

8. Show that if the plane face of the solid considered in Art. 50 instead of being kept at temperature zero is impervious to heat

$$u=\frac{1}{2a\sqrt{\pi t}}\int_0^\infty f(\lambda)\left(e^{-\frac{(\lambda-x)^2}{4a^2t}}+e^{-\frac{(\lambda+x)^2}{4a^2t}}\right)d\lambda. \qquad \text{v. (6) Art. 50.}$$

51. If the temperature of the plane face of the solid described in Art. 50 is a given function of the time and the initial temperature is zero, the solution of the problem can be obtained by a very ingenious method due to Riemann.

Here we have to solve the equation

$$D_tu=a^2D_x^2u \tag{1}$$

subject to the conditions

$$\left.\begin{aligned}u&=F(t)\quad\text{when}\quad x=0,\\ u&=0\qquad\;\;\text{"}\qquad t=0.\end{aligned}\right\} \tag{2}$$

We know that

$$u=\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t}}}e^{-\beta^2}d\beta$$

is a solution of (1), v. Ex. 1 Art. 50. It is easily shown that

$$u=\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t-c}}}e^{-\beta^2}d\beta, \tag{3}$$

where c is any constant, is a solution of (1).

For

$$D_tu=-\frac{2}{\sqrt{\pi}}\frac{x}{2a}\frac{1}{2(t-c)^{\frac{3}{2}}}e^{-\frac{x^2}{4a^2(t-c)}}=-\frac{x}{2a\sqrt{\pi}}(t-c)^{-\frac{3}{2}}e^{-\frac{x^2}{4a^2(t-c)}}$$

$$D_xu=\frac{2}{\sqrt{\pi}}\frac{1}{2a\sqrt{t-c}}e^{-\frac{x^2}{4a^2(t-c)}}$$

$$D_x^2u=-\frac{2}{\sqrt{\pi}}\frac{1}{2a\sqrt{t-c}}\frac{2x}{4a^2(t-c)}e^{-\frac{x^2}{4a^2(t-c)}}=-\frac{x}{2a^3\sqrt{\pi}}(t-c)^{-\frac{3}{2}}e^{-\frac{x^2}{4a^2(t-c)}}$$

and

$$D_tu=a^2D_x^2u.$$

Let $\phi(x, t)$ be a function of x and t which shall be equal to zero if t is negative and shall be equal to

$$1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2a\sqrt{t}}} e^{-\beta^2} d\beta$$

if t is equal to or greater than zero; so that if $x = 0$ $\phi(x, t) = 1$ and if $t = 0$ $\phi(x, t) = 0$.

We shall now attack the following problem, to solve equation (1) subject to the conditions

$$\begin{array}{llll} u = 0 & \text{if} & t = 0 & \\ u = F(0) & \text{"} & x = 0 & \text{and} \quad 0 < t < \tau \\ u = F(k\tau) & \text{"} & x = 0 & \text{"} \quad k\tau < t < (k+1)\tau, \end{array}$$

$= 0 \quad x = 0 \quad (k+1)\tau < t$

where k is any whole number and τ is any arbitrarily chosen interval of time.

If we form the value

$$u = F(k\tau)[\phi(x, t - k\tau) - \phi(x, t - (k+1)\tau)] \tag{4}$$

u will satisfy equation (1) since zero, unity and

$$\frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2a\sqrt{t - k\tau}}} e^{-\beta^2} d\beta$$

are values of u which satisfy (1). u will be zero if $t < k\tau$ by the definition of the function $\phi(x, t)$; if $x = 0$ $u = 0$ if $t > (k+1)\tau$ and $u = F(k\tau)$ if $k\tau < t < (k+1)\tau$.

Therefore

$$u = \sum_{k=0}^{k=\infty} F(k\tau)[\phi(x, t - k\tau) - \phi(x, t - (k+1)\tau)] \tag{5}$$

is the solution of the problem stated above.

(5) can be simplified somewhat from the consideration that for a given value of t $\phi(x, t - k\tau) = 0$ if $k\tau > t$. If, then, $n\tau$ is the greatest whole multiple of τ not exceeding t,

$$u = \sum_{k=0}^{k=n} F(k\tau)[\phi(x, t - k\tau) - \phi(x, t - (k+1)\tau)]. \tag{6}$$

If now we decrease τ indefinitely the limiting form of (6) will be the solution of the problem stated at the beginning of this article.

(6) may be written

$$u = \sum_{k=0}^{k=n} F(k\tau) \left[\frac{\phi(x, t - k\tau) - \phi(x, t - (k+1)\tau)}{\tau}\right]\tau \tag{7}$$

and if τ is indefinitely decreased the limiting form of (7) is

$$u = -\int_0^t F(\lambda) D_\lambda \phi(x, t-\lambda)\, d\lambda. \tag{8}$$

Since $t-\lambda$ is positive between the limits of integration

$$\phi(x, t-\lambda) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2a\sqrt{t-\lambda}}} e^{-\beta^2} d\beta,$$

and $$D_\lambda \phi(x, t-\lambda) = -\frac{x}{2a\sqrt{\pi}} e^{-\frac{x^2}{4a^2(t-\lambda)}} (t-\lambda)^{-\frac{3}{2}};$$

and (8) may be written

$$u = \frac{x}{2a\sqrt{\pi}} \int_0^t F(\lambda) e^{-\frac{x^2}{4a^2(t-\lambda)}} (t-\lambda)^{-\frac{3}{2}} d\lambda, \tag{9}$$

or if we let $$\beta = \frac{x}{2a\sqrt{t-\lambda}}$$

$$u = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\beta^2} F\left(t - \frac{x^2}{4a^2\beta^2}\right) d\beta. \tag{10}$$

EXAMPLES.

1. If $u = nt$ when $x = 0$ and $u = 0$ when $t = 0$

$$u = n\left(t + \frac{x^2}{2a^2}\right)\left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2a\sqrt{t}}} e^{-\beta^2} d\beta\right] - \frac{nx\sqrt{t}}{a\sqrt{\pi}} e^{-\frac{x^2}{4a^2t}}.$$

2. A thick iron slab is at the temperature zero throughout, one of its plane faces is then kept at the temperature 100° Centigrade for 5 minutes, then at the temperature zero for the next 5 minutes, then at the temperature 100° for the next 5 minutes, and then at the temperature zero. Required the temperature of a point in the slab 5 c.m. from the face at the expiration of 18 minutes. Given; $a^2 = .185$.

Ans., 20°.1.

3. If $u = F(t)$ when $x = 0$ and $u = f(x)$ when $t = 0$, then

$$u = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\beta^2} F\left(t - \frac{x^2}{4a^2\beta^2}\right) d\beta + \frac{1}{2a\sqrt{\pi t}} \int_0^{\infty} \left(e^{-\frac{(\lambda-x)^2}{4a^2t}} - e^{-\frac{(\lambda+x)^2}{4a^2t}}\right) f(\lambda)\, d\lambda.$$

v. (6) Art. 50.

4. If in Art. (51) $F(t)$ is a periodic function of the time of period T it can be expressed by a Fourier's series of the form

$$F(t) = \frac{1}{2}b_0 + \sum_{m=1}^{m=\infty}[a_m \sin mat + b_m \cos mat], \quad \text{where} \quad a = \frac{2\pi}{T},$$

or
$$F(t) = \frac{1}{2}b_0 + \sum_{m=1}^{m=\infty}\rho_m \sin(mat + \lambda_m),$$

where
$$\rho_m \cos\lambda_m = a_m \quad \text{and} \quad \rho_m \sin\lambda_m = b_m. \qquad \text{v. Art. 31 Ex. 3.}$$

Show that with this value of $F(t)$ (10) Art. 51 becomes

$$u = \frac{1}{\sqrt{\pi}}b_0\int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\beta^2}d\beta + \frac{2}{\sqrt{\pi}}\sum_{m=1}^{m=\infty}\rho_m\left[\sin(mat+\lambda_m)\int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\beta^2}\cos\frac{max^2}{4a^2\beta^2}d\beta\right.$$

$$\left. - \cos(mat+\lambda_m)\int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\beta^2}\sin\frac{max^2}{4a^2\beta^2}d\beta\right]$$

and that as t increases u approaches the value

$$u = \frac{1}{2}b_0 + \sum_{m=1}^{m=\infty}\rho_m e^{-\frac{x}{a}\sqrt{\frac{ma}{2}}}\sin\left(mat - \frac{x}{a}\sqrt{\frac{ma}{2}} + \lambda_m\right).$$

Given that

$$\int_0^\infty e^{-x^2}\sin\frac{b^2}{x^2}dx = \frac{\sqrt{\pi}}{2}e^{-b\sqrt{2}}\sin b\sqrt{2}; \quad \int_0^\infty e^{-x^2}\cos\frac{b^2}{x^2}dx = \frac{\sqrt{\pi}}{2}e^{-b\sqrt{2}}\cos b\sqrt{2}.$$

v. *Riemann, Lin. par. dif. gl.* § 54.

5. If we are dealing with a bar of small cross-section where the heat not only flows along the bar but at the same time escapes at the surface of the bar into air at the temperature zero we have to solve the differential equation

$$D_t u = a^2 D_x^2 u - b^2 u. \qquad \text{v. Fourier, Heat § 105.}$$

Show that for this case

$$u = e^{-(b^2+a^2\alpha^2)t}\sin\alpha x \quad \text{and} \quad u = e^{-(b^2+a^2\alpha^2)t}\cos\alpha x$$

are particular solutions, and that if $u = f(x)$ when $t = 0$

$$u = \frac{e^{-b^2t}}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(\lambda-x)^2}{4a^2t}}f(\lambda)d\lambda = \frac{e^{-b^2t}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\beta^2}f(x+2a\sqrt{t}.\beta)d\beta.$$

cf. (8) and (9) Art. 49.

If $u=0$ when $x=0$ and $u=f(x)$ when $t=0$

$$u=\frac{e^{-b^2t}}{\sqrt{\pi}}\left[\int\limits_{-\frac{x}{2a\sqrt{t}}}^{\infty}e^{-\beta^2}f(x+2a\sqrt{t}.\beta)d\beta-\int\limits_{\frac{x}{2a\sqrt{t}}}^{\infty}e^{-\beta^2}f(-x+2a\sqrt{t}.\beta)d\beta\right].$$

cf. (7) Art. 50.

If $u=-e^{-\frac{bx}{a}}$ when $t=0$ and $u=0$ when $x=0$

$$u=\frac{1}{\sqrt{\pi}}\left[e^{\frac{bx}{a}}\int\limits_{\frac{x}{2a\sqrt{t}}}^{\infty}e^{-(b\sqrt{t}+\beta)^2}d\beta-e^{-\frac{bx}{a}}\int\limits_{\frac{x}{2a\sqrt{t}}}^{\infty}e^{-(b\sqrt{t}-\beta)^2}d\beta\right],$$

and if $u=1$ when $x=0$ and $u=0$ when $t=0$ we have only to add $e^{-\frac{bx}{a}}$ to the second member of the last equation, since $u=e^{-\frac{bx}{a}}$ satisfies equation

$$D_t u = a^2 D_x^2 u - b^2 u.$$

If $u=F(t)$ when $x=0$ and $u=0$ when $t=0$ we can employ method of Art. 51.

$$\phi(x,t-\lambda)=e^{-\frac{bx}{a}}+\frac{1}{\sqrt{\pi}}\left[e^{\frac{bx}{a}}\int\limits_{\frac{x}{2a\sqrt{t-\lambda}}}^{\infty}e^{-(b\sqrt{t-\lambda}+\beta)^2}d\beta-e^{-\frac{bx}{a}}\int\limits_{\frac{x}{2a\sqrt{t-\lambda}}}^{\infty}e^{-(b\sqrt{t-\lambda}-\beta)^2}d\beta\right],$$

$$-D_\lambda\phi(x,t-\lambda)=\frac{x(t-\lambda)^{-\frac{3}{2}}}{2a\sqrt{\pi}}e^{-b^2(t-\lambda)-\frac{x^2}{4a^2(t-\lambda)}}$$

and

$$u=\frac{x}{2a\sqrt{\pi}}\int_0^t(t-\lambda)^{-\frac{3}{2}}e^{-b^2(t-\lambda)-\frac{x^2}{4a^2(t-\lambda)}}F(\lambda)d\lambda,$$

cf. (9) Art. 51,

or

$$u=\frac{2}{\sqrt{\pi}}\int\limits_{\frac{x}{2a\sqrt{t}}}^{\infty}e^{-\beta^2-\frac{b^2x^2}{4a^2\beta^2}}F\left(t-\frac{x^2}{4a^2\beta^2}\right)d\beta,$$

cf. (10) Art. 51.

If $F(t)$ is periodic and has the value taken in Ex. 4, show that the value approached by u as t increases is

$$u=\frac{1}{2}b_0e^{-\frac{bx}{a}}+\sum_{m=1}^{m=\infty}\rho_m e^{-\frac{p\sqrt{2}}{2a}x}\sin\left(m\alpha t-\frac{q\sqrt{2}}{2a}x+\lambda_m\right),$$

where $p=(b^2+\sqrt{b^4+m^2\alpha^2})^{\frac{1}{2}}$ and $q=(-b^2+\sqrt{b^4+m^2\alpha^2})^{\frac{1}{2}}$.

Given
$$\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}dx = \frac{\sqrt{\pi}}{2}e^{-2a}$$

$$\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\sin\frac{b^2}{x^2}dx = \frac{\sqrt{\pi}}{2}e^{-2c}\sin 2d$$

and
$$\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\cos\frac{b^2}{x^2}dx = \frac{\sqrt{\pi}}{2}e^{-2c}\cos 2d\,,$$

where

$$c = \frac{\sqrt{2}}{2}(a^2+\sqrt{a^4+b^4})^{\frac{1}{2}} \quad \text{and} \quad d = \frac{\sqrt{2}}{2}(-a^2+\sqrt{a^4+b^4})^{\frac{1}{2}}.$$

Ängstrom's method of determining the conductivity of a metal is based on the result just given (v. Phil. Mag. Feb. 1863), and is described by Sir Wm. Thomson (Encyc. Brit. Article "Heat") as by far the best that has yet been devised.

52. If u is a periodic function of the time when $x=0$ as in Art. 51 Ex. 4 and we are concerned with the limiting value approached by u as t increases we can avoid evaluating a complicated definite integral if we take the following course.

Since as we have seen in Art. 49 $u = e^{\beta t+\alpha x}$ is a solution of

$$D_t u = a^2 D_x^2 u \tag{1}$$

provided only that $\beta = a^2\alpha^2$ we have

$$u = e^{\beta t + \frac{x}{a}\sqrt{\beta}}$$

as a solution.

Replacing β by $\pm\beta i$ this becomes

$$u = e^{\pm\beta ti + \frac{x}{a}\sqrt{\beta}\sqrt{\pm i}}$$

or
$$u = e^{\pm\beta ti \pm\frac{x}{a}\sqrt{\frac{\beta}{2}}(1\pm i)}$$

since
$$\sqrt{i} = \pm\frac{1}{2}\sqrt{2}\,(1+i)$$

and
$$\sqrt{-i} = \pm\frac{1}{2}\sqrt{2}\,(1-i).$$

Hence

$$u = e^{-\frac{x}{a}\sqrt{\frac{\beta}{2}}}\sin\left(\beta t - \frac{x}{a}\sqrt{\frac{\beta}{2}}\right), \quad u = e^{-\frac{x}{a}\sqrt{\frac{\beta}{2}}}\cos\left(\beta t - \frac{x}{a}\sqrt{\frac{\beta}{2}}\right), \tag{2}$$

$$u = e^{\frac{x}{a}\sqrt{\frac{\beta}{2}}}\sin\left(\beta t + \frac{x}{a}\sqrt{\frac{\beta}{2}}\right), \quad u = e^{\frac{x}{a}\sqrt{\frac{\beta}{2}}}\cos\left(\beta t + \frac{x}{a}\sqrt{\frac{\beta}{2}}\right), \tag{3}$$

are particular solutions of (1).

From these we get readily

$$u = \rho_m e^{-\frac{x}{a}\sqrt{\frac{ma}{2}}} \sin\left(mat - \frac{x}{a}\sqrt{\frac{ma}{2}} + \lambda_m\right) \tag{4}$$

as a solution. (4) reduces to

$$u = \rho_m \sin(mat + \lambda_m) \quad \text{when} \quad x = 0$$

and to

$$u = \rho_m e^{-\frac{x}{a}\sqrt{\frac{ma}{2}}} \sin\left(\lambda_m - \frac{x}{a}\sqrt{\frac{ma}{2}}\right) \quad \text{when} \quad t = 0.$$

If we add a term which satisfies (1) and which is equal to zero when $x = 0$ and to $-\rho_m e^{-\frac{x}{a}\sqrt{\frac{ma}{2}}} \sin\left(\lambda_m - \frac{x}{a}\sqrt{\frac{ma}{2}}\right)$ when $t = 0$ (v. Art. 50) we shall have a solution of (1) which is zero when $t = 0$ and which is

$$\rho_m \sin(mat + \lambda_m) \quad \text{when} \quad x = 0.$$

The term in question approaches zero as t increases [v. (7) Art. 50] and we have at once the solution given in Art. 51 Ex. 4, as our required result.

EXAMPLE.

Show that $u = e^{\beta t + \alpha x}$ is a solution of $D_t u = a^2 D_x^2 u - b^2 u$ if $\beta = a^2\alpha^2 - b^2$, and hence that

$$u = e^{\beta t \pm \frac{x}{a}\sqrt{b^2+\beta}}, \quad u = e^{-\beta t \pm \frac{x}{a}\sqrt{b^2-\beta}}, \quad u = e^{\pm\beta t i \pm \frac{x}{a\sqrt{2}}(p+qi)},$$

$$u = e^{\pm\frac{px}{a\sqrt{2}}} \sin\left(\beta t \pm \frac{qx}{a\sqrt{2}}\right), \quad \text{and} \quad u = e^{\pm\frac{px}{a\sqrt{2}}} \cos\left(\beta t \pm \frac{qx}{a\sqrt{2}}\right),$$

where

$$p = [\sqrt{\beta^2 + b^4} + b^2]^{\frac{1}{2}} \quad \text{and} \quad q = [\sqrt{\beta^2 + b^4} - b^2]^{\frac{1}{2}},$$

are solutions. Hence

$$u = \rho_m e^{-\frac{px}{a\sqrt{2}}} \sin\left(\beta t - \frac{qx}{a\sqrt{2}} + \lambda_m\right)$$

is a solution.

If $\beta = ma$ this last result reduces to $u = \rho_m \sin(mat + \lambda_m)$ when $x = 0$ and by the reasoning of Art. 52 it must be the value u approaches as t increases if we have the same conditions as in the last part of Art. 51 Ex. 5.

53. The whole problem of the flow of heat is treated by Sir William Thomson (v. Math. and Phys. Papers, Vol. II), and other recent writers from a different and decidedly interesting point of view, which we shall briefly sketch in connection with the problem of *Linear Flow*.

Suppose we are dealing with a bar having a small cross-section and an adiathermanous surface, and take as our unit of heat the amount required to raise by a unit the temperature of a unit of length of the bar. If at a point of the bar a

quantity Q of heat is suddenly generated the point is called an *instantaneous heat source* of strength Q.

If the heat instead of being suddenly generated is generated gradually and at a rate that would give Q units of heat per unit of time the point is called a *permanent heat source* of strength Q.

The temperature at any point of the bar at any time due to an instantaneous source of strength Q at the point $x = \lambda$ is easily found by the aid of formula (8) Art. 49 as follows:—

If a quantity of heat Q is suddenly generated along the portion of the bar from $x = \lambda$ to $x = \lambda + \Delta\lambda$, where $\Delta\lambda$ is any arbitrary length, the temperature of that portion will be suddenly raised to $\frac{Q}{\Delta\lambda}$, and we shall have by (8) Art. 49

$$u = \frac{Q}{2a\sqrt{\pi t}}\,\frac{1}{\Delta\lambda}\int_{\lambda}^{\lambda+\Delta\lambda} e^{-\frac{(\lambda-x)^2}{4a^2t}}\,d\lambda \tag{1}$$

as the temperature of any point of the bar at any time t thereafter.

If now we write u equal to the limiting value approached by the second member of (1) as $\Delta\lambda$ is made to approach zero we get

$$u = \frac{Q}{2a\sqrt{\pi t}}\,e^{-\frac{(\lambda-x)^2}{4a^2t}} \tag{2}$$

as the solution for the case where we have an instantaneous source at the point $x = \lambda$.

It is to be observed that in (2) $u = 0$ when $t = 0$ and $u = \frac{Q}{2a\sqrt{\pi t}}$ when $x = \lambda$ and $t > 0$.

If we have several sources we have only to add the temperatures due to the separate sources.

Formula (8) Art. 49 may now be regarded as the solution for the case where we start with an instantaneous heat source of strength $f(\lambda)d\lambda$ in every element of length of the bar.

A source of strength $-Q$ is called a sink of strength Q; and (6) Art. 50 may be regarded as the solution for the case where we have at the start an instantaneous source of strength $f(\lambda)d\lambda$ in every element of the bar whose distance to the right of the origin is λ, and an instantaneous sink of strength $f(\lambda)d\lambda$ in every element of the bar whose distance to the left of the origin is λ.

If we have an instantaneous source at the origin (2) reduces to

$$u = \frac{Q}{2a\sqrt{\pi t}}\,e^{-\frac{x^2}{4a^2t}} \tag{3}$$

For a permanent source of constant strength Q at the origin (3) gives

$$u=\frac{Q}{2a\sqrt{\pi}}\int_0^t e^{-\frac{x^2}{4a^2(t-\tau)}}(t-\tau)^{-\frac{1}{2}}d\tau \tag{4}$$

and for a permanent source of variable strength $f(t)$

$$u=\frac{1}{2a\sqrt{\pi}}\int_0^t e^{-\frac{x^2}{4a^2(t-\tau)}}(t-\tau)^{-\frac{1}{2}}f(\tau)d\tau . \tag{5}$$

In (4) and (5) u obviously reduces to zero when $t=0$ and $x>0$, but its value when $x=0$ is not easily determined. We can avoid the difficulty by introducing the conception of a *doublet*.

54. If a source and a sink of equal strength Q are made to approach each other while Q multiplied by their distance apart is kept equal to a constant P the limiting state of things is said to be due to a *doublet* of strength P whose axis is tangent to the line of approach and points from sink to source. A *doublet* of strength $-P$ differs from a doublet of strength P only in that its axis has the opposite direction.

Let us find the temperature due to an instantaneous doublet of strength P placed at the origin. For a source of strength Q at $x=\eta$ and an equal sink at $x=-\eta$ we have

$$u=\frac{Q}{2a\sqrt{\pi t}}\left(e^{-\frac{(\eta-x)^2}{4a^2t}}-e^{-\frac{(\eta+x)^2}{4a^2t}}\right),$$

or if $2\eta Q=P$,

$$u=\frac{P}{4a\eta\sqrt{\pi t}}e^{-\frac{(\eta^2+x^2)}{4a^2t}}\left(e^{\frac{\eta x}{2a^2t}}-e^{-\frac{\eta x}{2a^2t}}\right)$$

$$=\frac{P}{2a\eta\sqrt{\pi t}}e^{-\frac{(\eta^2+x^2)}{4a^2t}}\sinh\frac{\eta x}{2a^2t}.$$

If η is made to approach zero

$$\text{limit}\left[\frac{1}{\eta}\sinh\frac{\eta x}{2a^2t}\right]=\frac{x}{2a^2t},$$

and

$$u=\frac{Px}{4a^3\sqrt{\pi t^3}}e^{-\frac{x^2}{4a^2t}} \tag{1}$$

is the solution for the temperature at any time and place due to an instantaneous doublet of strength P placed at the origin. For a doublet at any other point $x=\lambda$ we have

$$u=\frac{P(x-\lambda)}{4a^3\sqrt{\pi t^3}}e^{-\frac{(x-\lambda)^2}{4a^2t}}. \tag{2}$$

For a permanent doublet of constant strength P placed at the origin we have

$$u = \frac{Px}{4a^3\sqrt{\pi}}\int_0^t e^{-\frac{x^2}{4a^2(t-\tau)}}(t-\tau)^{-\frac{3}{2}}d\tau\,; \tag{3}$$

and for a permanent doublet of variable strength $f(t)$

$$u = \frac{x}{4a^3\sqrt{\pi}}\int_0^t e^{-\frac{x^2}{4a^2(t-\tau)}}(t-\tau)^{-\frac{3}{2}}f(\tau)d\tau\,, \tag{4}$$

or

$$u = \frac{1}{a^2\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\beta^2} f\left(t - \frac{x^2}{4a^2\beta^2}\right)d\beta \tag{5}$$

if $x > 0$, and

$$u = \frac{1}{a^2\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{-\infty} e^{-\beta^2} f\left(t - \frac{x^2}{4a^2\beta^2}\right)d\beta \tag{6}$$

if $x < 0$, if we let $\beta = \dfrac{x}{2a\sqrt{t-\tau}}$.

From (5) and (6) we see readily that $u = 0$ when $t = 0$ and that $u = \dfrac{f(t)}{2a^2}$ when $x = 0$ if we approach the origin from the right and that $u = -\dfrac{f(t)}{2a^2}$ when $x = 0$ if we approach the origin from the left.

If the point $x = 0$ is kept at the constant temperature b and we are concerned only with positive values of x we can get from (5) the solution given in Art. 50 Ex. 4 by supposing a permanent doublet of strength $2a^2b$ placed at the origin.

To solve the problem treated in Art. 51 we have only to suppose a permanent doublet of strength $2a^2F(t)$ placed at $x = 0$ and from (5) we get at once (10) Art. 51.

EXAMPLE.

Show that if $D_t u = a^2 D_x^2 u - b^2 u$ and an instantaneous source of strength Q is placed at $x = \lambda$

$$u = \frac{Q}{2a\sqrt{\pi t}}e^{-b^2t-\frac{(\lambda-x)^2}{4a^2t}}$$ v. Art. 51, Ex. 5.

Show that if an instantaneous doublet of strength P is placed at the point $x = 0$

$$u = \frac{Px}{4a^3\sqrt{\pi t^3}}e^{-b^2t-\frac{x^2}{4a^2t}}.$$

If a permanent doublet of strength $f(t)$ is placed at $x=0$

$$u=\frac{x}{4a^3\sqrt{\pi}}\int_0^t e^{-b^2(t-\tau)-\frac{x^2}{4a^2(t-\tau)}}(t-\tau)^{-\frac{3}{2}}f(\tau)d\tau$$

$$=\frac{1}{a^2\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\beta^2-\frac{b^2x^2}{4a^2\beta^2}}f\left(t-\frac{x^2}{4a^2\beta^2}\right)d\beta,$$

whence $u=0$ when $t=0$ and $x>0$ or $x<0$ and $u=\pm\frac{f(t)}{2a^2}$ when $x=0$.

Hence if we place at $x=0$ a permanent doublet of strength $2a^2F(t)$ we get the solution given in Art. 51 Ex. 5 for the case where $u=F(t)$ when $x=0$ and $u=0$ when $t=0$ provided we are concerned only with positive values of x.

If $F(t)=c$ this reduces to

$$u=\frac{2c}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\beta^2-\frac{b^2x^2}{4a^2\beta^2}}d\beta.$$

55. As another example of the use of Fourier's Integral we shall consider the transmission of a disturbance along a stretched elastic string.

Suppose we have a stretched elastic string so long that we need not consider what happens at its ends, that is so long that we may treat its length as infinite. Let the string be initially distorted into some given form and then released; to investigate its subsequent motion.

Let us take the position of equilibrium of the string as the axis of X and any given point as origin.

We have, then, to solve the differential equation

$$D_t^2y=a^2D_x^2y \tag{1}$$

[v. (VIII) Art. 1] subject to the conditions

$$y=f(x) \text{ when } t=0 \tag{2}$$

$$D_ty=0 \quad \text{"} \quad t=0. \tag{3}$$

As in Art. 8 we find

$$y=\cos\alpha(x\pm at) \quad \text{and} \quad y=\sin\alpha(x\pm at)$$

as particular solutions of (1).

From these we must build up a value that will reduce to

$$f(x)=\frac{1}{\pi}\int_0^{\infty}d\alpha\int_{-\infty}^{\infty}f(\lambda)\cos\alpha(\lambda-x)d\lambda \tag{4}$$

when $t=0$ and will at the same time satisfy (3).

$$y=\cos\alpha\lambda\cos\alpha(x+at)+\sin\alpha\lambda\sin\alpha(x+at)$$

or

$$y=\cos\alpha(\lambda-x-at)$$

is a solution of (1).

Hence
$$y=\frac{1}{\pi}\int_0^\infty d\alpha\int_{-\infty}^{\infty}f(\lambda)\cos\alpha(\lambda-x-at).d\lambda \tag{5}$$

is also a solution of (1).

(5) reduces to $y=f(x)$ when $t=0$ but it gives

$$D_t y=\frac{a}{\pi}\int_0^\infty \alpha d\alpha\int_{-\infty}^{\infty}f(\lambda)\sin\alpha(\lambda-x).d\lambda$$

when $t=0$ and consequently does not satisfy equation (3).

If in forming (5) we use $\cos\alpha(x-at)$ and $\sin\alpha(x-at)$ instead of $\cos\alpha(x+at)$ and $\sin\alpha(x+at)$ we get

$$y=\frac{1}{\pi}\int_0^\infty d\alpha\int_{-\infty}^{\infty}f(\lambda)\cos\alpha(\lambda-x+at).d\lambda \tag{6}$$

which is a solution of (1), and reduces to $y=f(x)$ when $t=0$, but it gives

$$D_t y=-\frac{a}{\pi}\int_0^\infty \alpha d\alpha\int_{-\infty}^{\infty}f(\lambda)\sin\alpha(\lambda-x).d\lambda$$

when $t=0$ and does not satisfy (3).

If, however, we take one-half the sum of the values of y in (5) and (6) we get

$$y=\frac{1}{2}\left[\frac{1}{\pi}\int_0^\infty d\alpha\int_{-\infty}^{\infty}f(\lambda)\cos\alpha(\lambda-x-at).d\lambda\right.$$
$$\left.+\frac{1}{\pi}\int_0^\infty d\alpha\int_{-\infty}^{\infty}f(\lambda)\cos\alpha(\lambda-x+at).d\lambda\right], \tag{7}$$

a solution of (1) which satisfies both (2) and (3), and is, therefore, our required solution.

This result can be very much simplified.

If we substitute $z=x+at$

$$\frac{1}{\pi}\int_0^\infty d\alpha\int_{-\infty}^{\infty}f(\lambda)\cos\alpha(\lambda-x-at).d\lambda$$
$$=\frac{1}{\pi}\int_0^\infty d\alpha\int_{-\infty}^{\infty}f(\lambda)\cos\alpha(\lambda-z).d\lambda=f(z)=f(x+at);$$

and in like manner we can show that

$$\frac{1}{\pi}\int_0^\infty da\int_{-\infty}^\infty f(\lambda)\cos a(\lambda - x + at).d\lambda = f(x - at).$$

Hence our solution becomes

$$y = \frac{1}{2}\left[f(x + at) + f(x - at)\right]. \tag{8}$$

This result is of great importance in the theory of elastic strings and it shows that the initial disturbance splits into two equal waves which run along the string, one to the right and the other to the left, with a uniform velocity a, and that there is nothing like a periodic motion or vibration of any sort unless the ends of the string produce some effect.

56. If the string is not initially distorted but starts from its position of equilibrium with a given initial velocity impressed upon each point we have to solve the equation

$$D_t^2 y = a^2 D_x^2 y \tag{1}$$

subject to the conditions

$$y = 0 \quad \text{when} \quad t = 0 \tag{2}$$

$$D_t y = F(x) \quad \text{"} \quad t = 0. \tag{3}$$

We get by the process used in Art. 55

$$y = \frac{1}{2\pi a}\int_0^\infty da\int_{-\infty}^\infty F(\lambda)\left[\frac{\sin a(\lambda - x + at)}{a} - \frac{\sin a(\lambda - x - at)}{a}\right]d\lambda$$

$$= \frac{1}{2\pi a}\int_{-\infty}^\infty F(\lambda)d\lambda\int_0^\infty\left[\frac{\sin a(\lambda - x + at)}{a} - \frac{\sin a(\lambda - x - at)}{a}\right]da;$$

but
$$\int_0^\infty\frac{\sin a(\lambda - x + at)}{a}da - \int_0^\infty\frac{\sin a(\lambda - x - at)}{a}da = \pi$$

if $x - at < \lambda < x + at$, and is equal to zero for all other values of λ; since

$$\int_0^\infty\frac{\sin mx}{x}dx = \frac{\pi}{2} \text{ if } m > 0$$

$$= -\frac{\pi}{2} \text{ if } m < 0$$

$$= 0 \text{ if } m = 0.$$

v. Int. Cal. Art. 92 (3).

Hence
$$y = \frac{1}{2a}\int_{x-at}^{x+at} F(\lambda)d\lambda \tag{4}$$

is our required solution.

EXAMPLES.

1. If the string is initially distorted and starts with initial velocity so that $y = f(x)$ and $D_t y = F(x)$ when $t = 0$

$$y = \frac{1}{2}\left[f(x + at) + f(x - at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} F(\lambda)d\lambda.$$

2. If the initial disturbance is caused by a blow, as from the hammer in a piano, which impresses upon all the points in a portion of the string of length c an equal transverse velocity b show that the front of the wave which will be seen to run to the left along the string will be a straight line having a slope equal to $\frac{b}{2a}$ and a length equal to $\frac{c}{2a}\sqrt{4a^2 + b^2}$. Of course a wave having a front of the same length with a slope equal to $-\frac{b}{2a}$ will be seen to run to the right along the string, and the effect of the two waves will be to lift the string bodily and permanently to a distance $\frac{bc}{2a}$ above its original position.

57. We shall now take up a few examples of the use of *Fourier's Series*. In the problem of Art. 7 let the temperature of the base of the plate be a given function of x, the other conditions remaining unchanged.

Since $$f(x) = \sum_{m=1}^{m=\infty}(a_m \sin mx)$$

where $$a_m = \frac{2}{\pi}\int_0^\pi f(\alpha)\sin m\alpha . d\alpha$$

we have $$u = \frac{2}{\pi}\sum_{m=1}^{m=\infty}\left[e^{-my}\sin mx\int_0^\pi f(\alpha)\sin m\alpha . d\alpha\right]. \tag{1}$$

If the breadth of the plate is a instead of π

$$u = \frac{2}{a}\sum_{m=1}^{m=\infty}\left[e^{-\frac{m\pi y}{a}}\sin\frac{m\pi x}{a}\int_0^a f(\lambda)\sin\frac{m\pi\lambda}{a}d\lambda\right]. \tag{2}$$

58. If the temperature of the base is unity and the breadth of the plate is π the solution is, as we have seen in Art. 7,

$$u = \frac{4}{\pi}\left[e^{-y}\sin x + \frac{1}{3}e^{-3y}\sin 3x + \frac{1}{5}e^{-5y}\sin 5x + \cdots\right]. \tag{1}$$

This series can be summed without difficulty. We have the development

$$\log(1 + z) = \frac{z}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

if the modulus of z is less than 1. Int. Cal. Art. 221 (4).

Hence $$\log(1-z) = -\frac{z}{1} - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \ldots$$

if mod. $z < 1$.

and $$\frac{1}{2}[\log(1+z) - \log(1-z)] = \frac{z}{1} + \frac{z^3}{3} + \frac{z^5}{5} + \ldots \tag{2}$$

if mod. $z < 1$.

But

$$\begin{aligned}\log(1+z) &= \log[1 + r(\cos\phi + i\sin\phi)] \\ &= \frac{1}{2}\log[(1+r\cos\phi)^2 + (r\sin\phi)^2] + i\tan^{-1}\frac{r\sin\phi}{1+r\cos\phi} \\ &= \frac{1}{2}\log(1+2r\cos\phi + r^2) + i\tan^{-1}\frac{r\sin\phi}{1+r\cos\phi},\end{aligned}$$

and

$$\log(1-z) = \frac{1}{2}\log(1-2r\cos\phi + r^2) - i\tan^{-1}\frac{r\sin\phi}{1-r\cos\phi},$$

[Int. Cal. Art. 33 (2)],
and (2) becomes

$$\frac{1}{2}\left[\frac{1}{2}\log\frac{1+2r\cos\phi+r^2}{1-2r\cos\phi+r^2} + i\tan^{-1}\frac{2r\sin\phi}{1-r^2}\right]$$
$$= \frac{r(\cos\phi + i\sin\phi)}{1} + \frac{r^3(\cos 3\phi + i\sin 3\phi)}{3} + \ldots \tag{3}$$

From (3) we get two equations

$$\frac{1}{4}\log\frac{1+2r\cos\phi+r^2}{1-2r\cos\phi+r^2} = \frac{r\cos\phi}{1} + \frac{r^3\cos 3\phi}{3} + \frac{r^5\cos 5\phi}{5} + \ldots \tag{4}$$

$$\frac{1}{2}\tan^{-1}\frac{2r\sin\phi}{1-r^2} = \frac{r\sin\phi}{1} + \frac{r^3\sin 3\phi}{3} + \frac{r^5\sin 5\phi}{5} + \ldots \tag{5}$$

both valid for all values of ϕ provided $r < 1$.

e^{-y} is less than 1 if y is positive.

Hence from (5)

$$\frac{e^{-y}\sin x}{1} + \frac{e^{-3y}\sin 3x}{3} + \frac{e^{-5y}\sin 5x}{5} + \ldots = \frac{1}{2}\tan^{-1}\frac{2e^{-y}\sin x}{1-e^{-2y}}$$
$$= \frac{1}{2}\tan^{-1}\frac{2\sin x}{e^{y}-e^{-y}} = \frac{1}{2}\tan^{-1}\frac{\sin x}{\sinh y},$$

and (1) may be written

$$u = \frac{2}{\pi}\tan^{-1}\frac{\sin x}{\sinh y}. \tag{6}$$

If we replace r by e^{-y} and ϕ by x in

$$\log\left[1+r(\cos\phi+i\sin\phi)\right]$$

it becomes
$$\log\left[1+e^{-y}\cos x+i\,e^{-y}\sin x\right]$$
or
$$\log\left[1+\cos z+i\sin z\right]$$
v. Int. Cal. Art. 35 (3) and (4)
a function of z as a whole; and

$$\log\left[1-r(\cos\phi+i\sin\phi)\right]$$

becomes
$$\log\left(1-\cos z-i\sin z\right);$$
hence by Int. Cal. Arts. 212 and 213,

$$\frac{1}{4}\log\frac{1+2e^{-y}\cos x+e^{-2y}}{1-2e^{-y}\cos x+e^{-2y}}\quad\text{and}\quad\frac{1}{2}\tan^{-1}\frac{2e^{-y}\sin x}{1-e^{-2y}}$$

or
$$\frac{1}{4}\log\frac{\cosh y+\cos x}{\cosh y-\cos x}\quad\text{and}\quad\frac{1}{2}\tan^{-1}\frac{\sin x}{\sinh y}$$

are conjugate functions, and

$$u_1=\frac{1}{\pi}\log\frac{\cosh y+\cos x}{\cosh y-\cos x}\tag{7}$$

is the solution for the problem where the isothermal lines are the lines of flow of the present problem and the lines of flow are the isothermal lines of the present problem.

For our problem, then, the isothermal lines are given by the equation

$$\frac{2}{\pi}\tan^{-1}\frac{\sin x}{\sinh y}=a$$

or
$$\frac{\sin x}{\sinh y}=\tan\frac{a\pi}{2}\tag{8}$$

and the lines of flow by

$$\frac{1}{\pi}\log\frac{\cosh y+\cos x}{\cosh y-\cos x}=b,$$

or
$$\frac{\cosh y+\cos x}{\cosh y-\cos x}=e^{\pi b}.\tag{9}$$

EXAMPLES.

1. If $D_x^2u+D_y^2u=0$, and $u=1$ when $y=0$, and $u=0$ when $x=0$ and when $x=a$,

$$u=\frac{4}{\pi}\left[e^{-\frac{\pi y}{a}}\sin\frac{\pi x}{a}+\frac{1}{3}e^{-\frac{3\pi y}{a}}\sin\frac{3\pi x}{a}+\frac{1}{5}e^{-\frac{5\pi y}{a}}\sin\frac{5\pi x}{a}+\cdots\right]$$

$$=\frac{2}{\pi}\tan^{-1}\frac{\sin\frac{\pi x}{a}}{\sinh\frac{\pi y}{a}}.$$

2. If $u=\phi(x)$ when $y=0$, $u=f(y)$ when $x=0$, and $u=F(y)$ when $x=a$

$$u=\frac{2}{a}\sum_{m=1}^{m=\infty}e^{-\frac{m\pi y}{a}}\sin\frac{m\pi x}{a}\int_0^a\phi(\lambda)\sin\frac{m\pi\lambda}{a}d\lambda$$

$$+\frac{1}{2a}\sin\frac{\pi x}{a}\int_0^\infty\left[\frac{1}{\cosh\frac{\pi}{a}(\lambda-y)-\cos\frac{\pi x}{a}}-\frac{1}{\cosh\frac{\pi}{a}(\lambda+y)-\cos\frac{\pi x}{a}}\right]f(\lambda)d\lambda$$

$$+\frac{1}{2a}\sin\frac{\pi x}{a}\int_0^\infty\left[\frac{1}{\cosh\frac{\pi}{a}(\lambda-y)+\cos\frac{\pi x}{a}}-\frac{1}{\cosh\frac{\pi}{a}(\lambda+y)+\cos\frac{\pi x}{a}}\right]F(\lambda)d\lambda$$

v. Art. 48, Exs. 4, 5, and 6.

59. If three sides of a plane rectangular sheet of conducting material be kept at potential zero and the value of the potential function at every point of the fourth side be given; to find the value of this potential function at any point of the sheet.

To formulate:—

$$D_x^2V+D_y^2V=0. \qquad (1)$$

$$V=0 \quad \text{when} \quad x=0, \qquad (2)$$

$$V=0 \quad \text{"} \quad x=a. \qquad (3)$$

$$V=0 \quad \text{"} \quad y=b. \qquad (4)$$

$$V=f(x) \quad \text{"} \quad y=0. \qquad (5)$$

Working as in Art. 48 we get

$$\frac{\sinh\frac{m\pi}{a}(b-y)}{\sinh\frac{m\pi b}{a}}\sin\frac{m\pi x}{a}$$

as a value of V which satisfies equations (1), (2), (3), and (4) if m is an integer. Therefore

$$V=\frac{2}{a}\sum_{m=1}^{m=\infty}\left[\frac{\sinh\frac{m\pi}{a}(b-y)}{\sinh\frac{m\pi b}{a}}\sin\frac{m\pi x}{a}\int_0^a f(\lambda)\sin\frac{m\pi\lambda}{a}d\lambda\right] \qquad (6)$$

is our required solution.

EXAMPLES.

1. If $f(x)=1$ Eq. (6) Art. 59 reduces to

$$V=\frac{4}{\pi}\left[\frac{\sinh\frac{\pi}{a}(b-y)}{\sinh\frac{\pi b}{a}}\sin\frac{\pi x}{a}+\frac{1}{3}\frac{\sinh\frac{3\pi}{a}(b-y)}{\sinh\frac{3\pi b}{a}}\sin\frac{3\pi x}{a}\right.$$

$$\left.+\frac{1}{5}\frac{\sinh\frac{5\pi}{a}(b-y)}{\sinh\frac{5\pi b}{a}}\sin\frac{5\pi x}{a}+\cdots\right]$$

2. If $V=0$ when $x=0$, $V=0$ when $x=a$, $V=0$ when $y=0$, [a]nd $V=F(x)$ when $y=b$, then

$$V=\frac{2}{a}\sum_{m=1}^{m=\infty}\left[\frac{\sinh\frac{m\pi y}{a}}{\sinh\frac{m\pi b}{a}}\sin\frac{m\pi x}{a}\int_0^a F(\lambda)\sin\frac{m\pi\lambda}{a}\,d\lambda\right].$$

3. If $F(x)=1$ the answer of Ex. 2 reduces to

$$V=\frac{4}{\pi}\left[\frac{\sinh\frac{\pi y}{a}}{\sinh\frac{\pi b}{a}}\sin\frac{\pi x}{a}+\frac{1}{3}\frac{\sinh\frac{3\pi y}{a}}{\sinh\frac{3\pi b}{a}}\sin\frac{3\pi x}{a}+\frac{1}{5}\frac{\sinh\frac{5\pi y}{a}}{\sinh\frac{5\pi b}{a}}\sin\frac{5\pi x}{a}+\cdots\right].$$

4. If $V=0$ when $x=0$, $V=0$ when $x=a$, $V=f(x)$ when $y=0$; [a]nd $V=F(x)$ when $y=b$, then

$$V=\frac{2}{a}\sum_{m=1}^{m=\infty}\left[\sin\frac{m\pi x}{a}\left(\frac{\sinh\frac{m\pi}{a}(b-y)}{\sinh\frac{m\pi b}{a}}\int_0^a f(\lambda)\sin\frac{m\pi\lambda}{a}\,d\lambda\right.\right.$$

$$\left.\left.+\frac{\sinh\frac{m\pi y}{a}}{\sinh\frac{m\pi b}{a}}\int_0^a F(\lambda)\sin\frac{m\pi\lambda}{a}\,d\lambda\right)\right].$$

5. If $f(x)=F(x)$ the answer of Ex. 4 reduces to

$$V=\frac{2}{a}\sum_{m=1}^{m=\infty}\left[\frac{\cosh\frac{m\pi}{a}\left(\frac{b}{2}-y\right)}{\cosh\frac{m\pi b}{2a}}\sin\frac{m\pi x}{a}\int_0^a f(\lambda)\sin\frac{m\pi\lambda}{a}\,d\lambda\right].$$

6. If $f(x)=F(x)=1$ the answer of Ex. 5 reduces to

$$V=\frac{4}{\pi}\left[\frac{\cosh\frac{\pi}{a}\left(\frac{b}{2}-y\right)}{\cosh\frac{\pi b}{2a}}\sin\frac{\pi x}{a}+\frac{1}{3}\frac{\cosh\frac{3\pi}{a}\left(\frac{b}{2}-y\right)}{\cosh\frac{3\pi b}{2a}}\sin\frac{3\pi x}{a}\right.$$

$$\left.+\frac{1}{5}\frac{\cosh\frac{5\pi}{a}\left(\frac{b}{2}-y\right)}{\cosh\frac{5\pi b}{2a}}\sin\frac{5\pi x}{a}+\cdots\right].$$

7. If $V=f(x)$ when $y=0$, $V=F(x)$ when $y=b$, $V=\phi(y)$ when $x=0$, and $V=\chi(y)$ when $x=a$, then

$$V=\frac{2}{a}\sum_{m=1}^{m=\infty}\left[\sin\frac{m\pi x}{a}\left(\frac{\sinh\frac{m\pi}{a}(b-y)}{\sinh\frac{m\pi b}{a}}\int_0^a f(\lambda)\sin\frac{m\pi\lambda}{a}d\lambda\right.\right.$$

$$\left.\left.+\frac{\sinh\frac{m\pi y}{a}}{\sinh\frac{m\pi b}{a}}\int_0^a F(\lambda)\sin\frac{m\pi\lambda}{a}d\lambda\right)\right]$$

$$+\frac{2}{b}\sum_{m=1}^{m=\infty}\left[\sin\frac{m\pi y}{b}\left(\frac{\sinh\frac{m\pi}{b}(a-x)}{\sinh\frac{m\pi a}{b}}\int_0^b\phi(\lambda)\sin\frac{m\pi\lambda}{b}d\lambda\right.\right.$$

$$\left.\left.+\frac{\sinh\frac{m\pi x}{b}}{\sinh\frac{m\pi a}{b}}\int_0^b\chi(\lambda)\sin\frac{m\pi\lambda}{b}d\lambda\right)\right].$$

8. If $f(x)=\phi(y)=0$ and $F(x)=\chi(y)=1$ the answer of Ex. 7 may be reduced to

$$V=\frac{2}{\pi}\left[\frac{\pi y}{2b}-\frac{\sinh\frac{\pi}{b}\left(\frac{a}{2}-x\right)}{\sinh\frac{\pi a}{2b}}\sin\frac{\pi y}{b}+\frac{1}{2}\frac{\cosh\frac{2\pi}{b}\left(\frac{a}{2}-x\right)}{\cosh\frac{2\pi a}{2b}}\sin\frac{2\pi y}{b}\right.$$

$$\left.-\frac{1}{3}\frac{\sinh\frac{3\pi}{b}\left(\frac{a}{2}-x\right)}{\sinh\frac{3\pi a}{2b}}\sin\frac{3\pi y}{b}+\frac{1}{4}\frac{\cosh\frac{4\pi}{b}\left(\frac{a}{2}-x\right)}{\cosh\frac{4\pi a}{2b}}\sin\frac{4\pi y}{b}-\cdots\right].$$

9. Find the temperature of the middle point of a thin square plate whose faces are impervious to heat; 1st, when three edges are kept at the temperature 0° and the fourth edge at the temperature 100°; 2d, when two opposite edges are kept at the temperature 0° and the other two at the temperature 100°; 3d, when two adjacent edges are kept at the temperature 0° and the other edges at the temperature 100°. See examples 3, 6, and 8.

Ans., (1) 25°; (2) 50°; (3) 50°.

60. Let us pass on to the consideration of the flow of heat in one dimension. Suppose that we have an infinite solid with two parallel plane faces whose distance apart is c.

Take the origin in one face and the axis of X perpendicular to the faces. Let the initial temperature be any given function of x and let the two faces be kept at the constant temperature zero; to find the temperature at any point of the slab at any time.

We have to solve the equation

$$D_t u = a^2 D_x^2 u \tag{1}$$

subject to the conditions

$$u = 0 \quad \text{when} \quad x = 0 \tag{2}$$

$$u = 0 \quad \text{"} \quad x = c \tag{3}$$

$$u = f(x) \quad \text{"} \quad t = 0. \tag{4}$$

In Art. 49 we have found

$$u = e^{-a^2\alpha^2 t} \sin \alpha x$$

and

$$u = e^{-a^2\alpha^2 t} \cos \alpha x$$

as particular solutions of (1).

$u = e^{-a^2\alpha^2 t}\sin \alpha x$ satisfies (2) whatever value is given to α. It satisfies (3) if $\alpha = \dfrac{m\pi}{c}$ provided m is an integer. Let us try to build a value of u out of terms of the form $Ae^{-\frac{a^2 m^2\pi^2 t}{c^2}} \sin \dfrac{m\pi x}{c}$ which shall satisfy (4).

We have

$$f(x) = \frac{2}{c}\sum_{m=1}^{m=\infty}\left[\sin\frac{m\pi x}{c}\int_0^c f(\lambda)\sin\frac{m\pi\lambda}{c}\,d\lambda\right]. \tag{5}$$

$$u = \frac{2}{c}\sum_{m=1}^{m=\infty}\left[e^{-\frac{a^2m^2\pi^2 t}{c^2}}\sin\frac{m\pi x}{c}\int_0^c f(\lambda)\sin\frac{m\pi\lambda}{c}\,d\lambda\right], \tag{6}$$

reduces to (5) when $t = 0$ and is our required solution.

EXAMPLES.

1. If $f(\lambda)=b$, a constant, (6) Art. 60 reduces to

$$u=\frac{4b}{\pi}\left[e^{-\frac{a^2\pi^2t}{c^2}}\sin\frac{\pi x}{c}+\frac{1}{3}e^{-\frac{9a^2\pi^2t}{c^2}}\sin\frac{3\pi x}{c}+\frac{1}{5}e^{-\frac{25a^2\pi^2t}{c^2}}\sin\frac{5\pi x}{c}+\cdots\right].$$

2. An iron slab 10 cm. thick is placed between and in contact with two other iron slabs each 10 cm. thick. The temperature of the middle slab is at first 100° throughout, and of the outside slabs 0° throughout. The outer faces of the outside slabs are kept at the temperature 0°. Required the temperature of a point in the middle of the middle slab fifteen minutes after the slabs have been placed in contact. (Given $a^2=0.185$ in C.G.S. units. *Ans.*, 10°.3.

3. Two iron slabs each 20 cm. thick one of which is at the temperature 0° and the other at the temperature 100° throughout, are placed together face to face, and their outer faces are kept at the temperature 0°. Find the temperature of a point in their common face and of points 10 cm. from the common face fifteen minutes after the slabs have been put together.

Ans., 22°.8; 15°.1; 17°.2.

4. One face of an iron slab 40 cm. thick is kept at the temperature 0° and the other face at the temperature 100° until the permanent state of temperatures is set up. Each face is then kept at the temperature 0°. Required the temperature of a point in the middle of the slab, and of points 10 cm. from the faces fifteen minutes after the cooling has begun.

Ans., 22°.8; 15°.6; 16°.7.

61. If the faces of the slab treated in Art. 60 instead of being kept at the temperature zero are rendered impervious to heat, the solution of the problem is easy.

In this case we have to solve the equation

$$D_tu=a^2D_x^2u$$

subject to the conditions

$$D_xu=0 \text{ when } x=0$$

$$D_xu=0 \quad " \quad x=c$$

$$u=f(x) \quad " \quad t=0.$$

We have only to use the particular solution

$$u=e^{-a^2\alpha^2t}\cos\alpha x$$

as we used

$$u=e^{-a^2\alpha^2t}\sin\alpha x$$

in Art. 60. We get

$$u=\frac{2}{c}\left[\frac{1}{2}\int_0^c f(\lambda)d\lambda+\sum_{m=1}^{m=\infty}\left(e^{-\frac{m^2a^2\pi^2t}{c^2}}\cos\frac{m\pi x}{c}\int_0^c f(\lambda)\cos\frac{m\pi\lambda}{c}d\lambda\right)\right]. \tag{1}$$

EXAMPLES.

1. Solve example 2 Art. 60 supposing that the outer surfaces are blanketed after the slabs are placed together so that heat can neither enter nor escape. Find in addition the temperature of the outer surfaces fifteen minutes after the slabs are placed in contact. *Ans.*, 33°.3; 33°.3.

2. Solve example 3 Art. 60 on the hypothesis just stated, getting in addition the temperatures of points on the outer surfaces.

Ans., 50°; 33°.9; 66°.1; 27°.2; 72°.8.

3. Solve example 4 Art. 60 supposing that heat neither enters nor escapes at the outer surfaces after the permanent state of temperatures has been set up. Find also the temperatures of points in the outer surfaces.

Ans., 50°; 39°.7; 60°.3; 35°.5; 64°.5.

4. Show that if $u=0$ when $x=0$, $D_x u=0$ when $x=c$, and $u=f(x)$ when $t=0$,

$$u=\frac{2}{c}\sum_{m=0}^{m=\infty}\left(e^{-\frac{(2m+1)^2a^2\pi^2t}{4c^2}}\sin\frac{(2m+1)\pi x}{2c}\int_0^c f(\lambda)\sin\frac{(2m+1)\pi\lambda}{2c}\,d\lambda\right).$$

Suggestion: Assume $u=0$ when $x=2c$ and $f(2c-x)=f(x)$, and see (6) Art. 60.

62. If the temperature of the right-hand face of the slab considered in Art. 60 is a constant γ instead of zero we have only to add to the second member of (6) Art. 60 a term u_1 which shall satisfy the conditions

$$D_t u_1 = a^2 D_x^2 u_1 \qquad (1)$$

$$u_1 = 0 \quad \text{when} \quad x=0 \qquad (2)$$

$$u_1 = 0 \quad \text{"} \quad t=0 \qquad (3)$$

$$u_1 = \gamma \quad \text{"} \quad x=c. \qquad (4)$$

$u_1=\frac{\gamma x}{c}$ obviously satisfies (1), (2), and (4); to make it satisfy (3) as well we must add a term u_2 which shall be equal to zero when $x=0$ and when $x=c$ and to $-\frac{\gamma x}{c}$ when $t=0$, while always satisfying (1). It is given immediately by (6) Art. 60 and is

$$u_2=-\frac{2\gamma}{c^2}\sum_{m=1}^{m=\infty}\left(e^{-\frac{m^2a^2\pi^2t}{c^2}}\sin\frac{m\pi x}{c}\int_0^c\lambda\sin\frac{m\pi\lambda}{c}\,d\lambda\right). \qquad (5)$$

$$\int_0^c\lambda\sin\frac{m\pi\lambda}{c}\,d\lambda=-\frac{c^2}{m\pi}\cos m\pi=(-1)^{m+1}\frac{c^2}{m\pi},$$

and
$$u_2 = \frac{2\gamma}{\pi}\sum_{m=1}^{m=\infty}\left(\frac{(-1)^m}{m}e^{-\frac{m^2a^2\pi^2t}{c^2}}\sin\frac{m\pi x}{c}\right). \tag{6}$$

Hence
$$u_1 = \gamma\left[\frac{x}{c} + \frac{2}{\pi}\sum_{m=1}^{m=\infty}\left(\frac{(-1)^m}{m}e^{-\frac{m^2a^2\pi^2t}{c^2}}\sin\frac{m\pi x}{c}\right)\right]. \tag{7}$$

If the left-hand face of the slab considered in Art. 60 is to be kept at a constant temperature β and the right-hand face at the temperature zero we can get the term u_3 which must be added to the second member of (6) Art. 60 by replacing γ by β and x by $c-x$ in (7). We then have

$$u_3 = \beta\left[\frac{c-x}{c} - \frac{2}{\pi}\sum_{m=1}^{m=\infty}\left(\frac{1}{m}e^{-\frac{m^2a^2\pi^2t}{c^2}}\sin\frac{m\pi x}{c}\right)\right]. \tag{8}$$

EXAMPLES.

1. Show that if $u=\beta$ when $x=0$, $u=\gamma$ when $x=c$, and $u=f(x)$ when $t=0$

$$u=\beta+(\gamma-\beta)\left[\frac{x}{c}+\frac{2}{\pi}\sum_{m=1}^{m=\infty}\left(\frac{(-1)^m}{m}e^{-\frac{m^2a^2\pi^2t}{c^2}}\sin\frac{m\pi x}{c}\right)\right]$$

$$+\frac{2}{c}\sum_{m=1}^{m=\infty}\left(e^{-\frac{m^2a^2\pi^2t}{c^2}}\sin\frac{m\pi x}{c}\int_0^c[f(\lambda)-\beta]\sin\frac{m\pi\lambda}{c}d\lambda\right).$$

2. Show that if $u=\beta$ when $x=0$, $u=0$ when $t=0$, and $D_x u=0$ when $x=c$

$$u=\beta\left[1-\frac{4}{\pi}\sum_{m=0}^{m=\infty}\left(\frac{1}{2m+1}e^{-\frac{(2m+1)^2a^2\pi^2t}{4c^2}}\sin\frac{(2m+1)\pi x}{2c}\right)\right]$$

$$=\beta\left[1-\frac{4}{\pi}\left(e^{-\frac{a^2\pi^2t}{4c^2}}\sin\frac{\pi x}{2c}+\frac{1}{3}e^{-\frac{9a^2\pi^2t}{4c^2}}\sin\frac{3\pi x}{2c}+\frac{1}{5}e^{-\frac{25a^2\pi^2t}{4c^2}}\sin\frac{5\pi x}{2c}+\cdots\right)\right].$$

63. If the temperature of the right-hand face of the slab just considered is a function of the time instead of a constant and the temperature of the left-hand face is zero the problem can be solved by a method nearly identical with that of Art. 51.

Let $\phi(x,t)$ be a function of x and t which shall be zero if t is less than zero and shall be equal to

$$\frac{x}{c} + \frac{2}{\pi}\sum_{m=1}^{m=\infty}\left(\frac{(-1)^m}{m}e^{-\frac{m^2a^2\pi^2t}{c^2}}\sin\frac{m\pi x}{c}\right)$$

[v. (7) Art. 62] if t is equal to or greater than zero. So that

$$\begin{aligned}
&\phi(x,t)=0 &&\text{if} && t<0 \\
&\phi(x,t)=0 &&\text{"} && t=0 \quad \text{unless} \quad x=c \\
&\phi(x,t)=1 &&\text{"} && t=0 \quad \text{and} \quad x=c \\
&\phi(x,t)=1 &&\text{"} && x=c \\
&\phi(x,t)=0 &&\text{"} && x=0\,.
\end{aligned}$$

Precisely as in Art. 51 we get

$$u = \underset{\tau \doteq 0}{\text{limit}}\sum_{k=0}^{k=n}\left[F(k\tau)\frac{[\phi(x,t-k\tau)-\phi(x,t-(k+1)\tau)]\tau}{\tau}\right] \tag{1}$$

as the required solution of our problem, n being as in Art. 51 the largest integer in $\frac{t}{\tau}$ where t is any given value of the time.

On our hypothesis the last term of (1), that is, $-F(n\tau)\phi[x,t-(n+1)\tau]=0$; the next to the last term $F(n\tau)\phi(x,t-n\tau)$ has for its limiting value

$$F(t)\phi(x,0) = F(t)\left[\frac{x}{c} + \frac{2}{\pi}\sum_{m=1}^{m=\infty}\left(\frac{(-1)^m}{m}\sin\frac{m\pi x}{c}\right)\right],$$

while as in Art. 51 the limiting value of the rest of the sum is

$$-\int_0^t F(\lambda)\,D_\lambda\phi(x,t-\lambda)d\lambda.$$

$$D_\lambda\phi(x,t-\lambda) = \frac{2a^2\pi}{c^2}\sum_{m=1}^{m=\infty}\left[(-1)^m m e^{-\frac{m^2a^2\pi^2}{c^2}(t-\lambda)}\sin\frac{m\pi x}{c}\right].$$

Hence

$$u = F(t)\left[\frac{x}{c} + \frac{2}{\pi}\sum_{m=1}^{m=\infty}\left(\frac{(-1)^m}{m}\sin\frac{m\pi x}{c}\right)\right]$$

$$-\frac{2a^2\pi}{c^2}\sum_{m=1}^{m=\infty}\left((-1)^m m\sin\frac{m\pi x}{c}\int_0^t F(\lambda)e^{-\frac{m^2a^2\pi^2}{c^2}(t-\lambda)}d\lambda\right),$$

$$u=\frac{x}{c}F(t)+\frac{2}{\pi}\sum_{m=1}^{m=\infty}\left[\frac{(-1)^m}{m}\sin\frac{m\pi x}{c}\left(F(t)\right.\right.$$

$$\left.\left.-\frac{m^2a^2\pi^2}{c^2}\int_0^t F(\lambda)e^{-\frac{m^2a^2\pi^2}{c^2}(t-\lambda)}d\lambda\right)\right]. \qquad (2)$$

If we substitute $\beta=\frac{m^2a^2\pi^2}{c^2}(t-\lambda)$ we get

$$u=\frac{x}{c}F(t)+\frac{2}{\pi}\sum_{m=1}^{m=\infty}\left[\frac{(-1)^m}{m}\sin\frac{m\pi x}{c}\left(F(t)-\int_0^{\frac{m^2a^2\pi^2t}{c^2}}e^{-\beta}F\left(t-\frac{\beta c^2}{m^2a^2\pi^2}\right)d\beta\right)\right]. \qquad (3)$$

EXAMPLES.

1. If the temperature of the left-hand face is a function of t and the temperature of the right-hand face is zero and the initial temperature is zero

$$u=\left(1-\frac{x}{c}\right)F(t)-\frac{2}{\pi}\sum_{m=1}^{m=\infty}\left[\frac{1}{m}\sin\frac{m\pi x}{c}\left(F(t)-\int_0^{\frac{m^2a^2\pi^2t}{c^2}}e^{-\beta}F\left(t-\frac{\beta c^2}{m^2a^2\pi^2}\right)d\beta\right)\right].$$

2. If the temperature of the left-hand face is a function of t, the initial temperature is zero, and the right-hand face is impervious to heat,

$$u=F(t)-\frac{4}{\pi}\sum_{m=0}^{m=\infty}\left[\frac{1}{2m+1}\sin\frac{(2m+1)\pi x}{2c}\left(F(t)\right.\right.$$

$$\left.\left.-\frac{(2m+1)^2a^2\pi^2}{4c^2}\int_0^t F(\lambda)e^{-\frac{(2m+1)^2a^2\pi^2}{4c^2}(t-\lambda)}d\lambda\right)\right].$$

3. If in Arts. 60–63 we are dealing with a bar of small cross-section and of length c and heat is radiating from the surface of the bar into air at the temperature zero so that $D_t u=a^2D_x^2u-b^2u$, show that (a) the second members of (6) Art. 60 and (1) Art. 61 must be multiplied by e^{-b^2t}; (b) equation (7) Art. 62 becomes

$$u_1=\gamma\left\{\frac{\sinh\frac{bx}{a}}{\sinh\frac{bc}{a}}+2a^2\pi e^{-b^2t}\sum_{m=1}^{m=\infty}\left[(-1)^m\frac{m}{b^2c^2+m^2a^2\pi^2}e^{-\frac{m^2a^2\pi^2t}{c^2}}\sin\frac{m\pi x}{c}\right]\right\};$$

(c) equation (2) Art. 63 becomes

$$u = \frac{\sinh \frac{bx}{a}}{\sinh \frac{bc}{a}} F(t) + 2a^2\pi \sum_{m=1}^{m=\infty} \left\{ \frac{(-1)^m m}{b^2c^2 + m^2a^2\pi^2} \sin \frac{m\pi x}{c} \left[F(t) \right.\right.$$

$$\left.\left. - \frac{b^2c^2 + m^2a^2\pi^2}{c^2} \int_0^t e^{-\frac{b^2c^2+m^2a^2\pi^2}{c^2}(t-\lambda)} F(\lambda) d\lambda \right] \right\}.$$

64. The problem of the motion of a finite stretched elastic string of length l fastened at the ends and distorted at first into some given curve $y = f(x)$, and then allowed to swing, has been treated and partially solved in Art. 8.

The complete solution is easily seen to be

$$y = \frac{2}{l} \sum_{m=1}^{m=\infty} \sin \frac{m\pi x}{l} \cos \frac{m\pi at}{l} \int_0^l f(\lambda) \sin \frac{m\pi\lambda}{l} d\lambda. \qquad (1)$$

The second member of (1) is a periodic function of t having the period $\frac{2l}{a}$. The motion, then, unlike that in the case of an infinite string (Art. 55) is a true vibration, a periodic motion. The period $\frac{2l}{a}$ is the time it takes a disturbance to travel twice the length of the string (v. Art. 55).

A careful examination of (1) will show that the actual motion is a good deal like that in the case considered in Art. 55. The original disturbance breaks up into two waves one of which runs to the right until it reaches the end of the string and is then reflected, and runs back to the left or the under side of the string, while the other wave runs to the left and is reflected at the left-hand end of the string and runs back to the right under the string and is again reflected, runs back to the left over the string and so on indefinitely.

If the curve into which the string is distorted at the start is of the form $y = b \sin \frac{m\pi x}{l}$ the solution is

$$y = b \sin \frac{m\pi x}{l} \cos \frac{m\pi at}{l}. \qquad (2)$$

No matter what value t may have the curve is always of the form

$$y = A \sin \frac{m\pi x}{l};$$

that is, for different values of t we have a set of sine curves differing only in the amplitude and not at all in the period of the curve. In this case either the whole string if $m = 1$, or each mth of the string if m is not equal to one, rises and falls, and there is no apparent onward motion. When this is the case we are said to have a *steady* vibration.

If $m=1$ we get steady motion of the string as a whole and if the vibration is rapid enough to give a musical note the note is said to be the pure *fundamental* note of the string. If $m=2$ the vibration is twice as rapid as when $m=1$, the middle point of the string does not move and is called a *node*, the two halves of the string are in opposite phases of vibration at any instant, and the note given is an octave higher than the fundamental note and is called its pure *first harmonic*.

If $m=3$ the vibration is three times as rapid as in the first case, there are two nodes $x=\frac{l}{3}$ and $x=\frac{2l}{3}$, and the note is the pure *second harmonic* of the fundamental note.

For any value of m the vibration is m times as rapid as when $m=1$, there are $m-1$ nodes at the points $x=\frac{l}{m}$, $x=\frac{2l}{m}$, ... $x=\frac{m-1}{m}l$, and we get the $m-1$st harmonic of the fundamental note.

It is clear from (1) that no matter what the original form of the string the resulting vibration can be regarded as a combination of steady vibrations each of which alone would give the fundamental note of the string or one of its harmonics, and that the complex note resulting is really a concord of the fundamental note and some of its harmonics.

A finely trained ear can often recognize in a complex note the fundamental note of the string and some of its harmonics and is capable of analyzing a complex note into its component pure notes precisely as Fourier's Theorem enables us to analyze the complex function representing the initial form of the string into the simpler sine-functions which must be combined to form it.

EXAMPLES.

1. Show that if a point whose distance from the end of a harp string is $\frac{1}{n}$th the length of the string is drawn aside by the player's finger to a distance b from its position of equilibrium and then released, the form of the vibrating string at any instant is given by the equation

$$y=\frac{2bn^2}{(n-1)\pi^2}\sum_{m=1}^{m=\infty}\left(\frac{1}{m^2}\sin\frac{m\pi}{n}\sin\frac{m\pi x}{l}\cos\frac{m\pi at}{l}\right).$$

Show from this that all the harmonics of the fundamental note of the string which correspond to forms of vibration having nodes at the point drawn aside by the finger will be wanting in the complex note actually sounded.

2. If a stretched string starts from its position of equilibrium, each of its points having a given initial velocity, so that we have

$$
\begin{array}{rcl}
y = 0 & \text{when} & t = 0 \\
D_t y = F(x) & `` & t = 0 \\
y = 0 & `` & x = 0 \\
y = 0 & `` & x = l,
\end{array}
$$

the solution of the problem of its vibration is easy and gives

$$y = \frac{2}{a\pi} \sum_{m=1}^{m=\infty} \left(\frac{1}{m} \sin \frac{m\pi x}{l} \sin \frac{m\pi a t}{l} \int_0^l F(\lambda) \sin \frac{m\pi\lambda}{l} d\lambda \right).$$

3. Write down the solution for the case where the string is initially distorted and each point has a given initial velocity.

65. If we do not neglect the resistance of the air in the problem of the vibration of a stretched string the differential equation is rather more complicated and the solution is not so easily obtained. The equation is given as (IX) Art. 1.

Let us solve the problem for the case where there is no initial velocity.

Here we have $$D_t^2 y + 2kD_t y = a^2 D_x^2 y. \qquad (1)$$

$$
\begin{array}{rcll}
y = 0 & \text{when} & x = 0 & (2) \\
y = 0 & `` & x = l & (3) \\
y = f(x) & `` & t = 0 & (4) \\
D_t y = 0 & `` & t = 0. & (5)
\end{array}
$$

We get particular solutions of (1) in the usual way. Assume $y = e^{\alpha x + \beta t}$ and substitute in (1). We have

$$\beta^2 + 2k\beta = a^2\alpha^2$$

as the only necessary relation between β and α. This gives

$$\beta = -k \pm \sqrt{a^2\alpha^2 + k^2}.$$

Hence $$y = e^{\alpha x - kt \pm t\sqrt{a^2\alpha^2 + k^2}} \qquad (6)$$

is a solution of (1) no matter what the value of α.

To throw it into Trigonometric form replace α by αi, and since in actual problems k, which is proportional to the resistance, is very small, take -1 out as a factor of the radical. We have

$$y = e^{-kt} e^{(\alpha x \pm t\sqrt{a^2\alpha^2 - k^2})i}.$$

Since α may be positive or negative we can get

$$y = e^{-kt}\sin\left(\alpha x \pm t\sqrt{a^2\alpha^2 - k^2}\right)$$

and

$$y = e^{-kt}\cos\left(\alpha x \pm t\sqrt{a^2\alpha^2 - k^2}\right)$$

as solutions of (1), or by combining these

$$y = e^{-kt}\sin\alpha x\cos t\sqrt{a^2\alpha^2 - k^2} \tag{7}$$

$$y = e^{-kt}\sin\alpha x\sin t\sqrt{a^2\alpha^2 - k^2} \tag{8}$$

$$y = e^{-kt}\cos\alpha x\cos t\sqrt{a^2\alpha^2 - k^2} \tag{9}$$

$$y = e^{-kt}\cos\alpha x\sin t\sqrt{a^2\alpha^2 - k^2} \tag{10}$$

(7) and (8) satisfy (1) and (2) for all values of α. They satisfy (3) if $\alpha = \dfrac{m\pi}{l}$. Let us see if out of them we cannot build up a value that will satisfy (4) and (5) as well.

$$f(x) = \frac{2}{l}\sum_{m=1}^{m=\infty}\left(\sin\frac{m\pi x}{l}\int_0^l f(\lambda)\sin\frac{m\pi\lambda}{l}\,d\lambda\right). \tag{11}$$

$$y = \frac{2}{l}e^{-kt}\sum_{m=1}^{m=\infty}\left(\sin\frac{m\pi x}{l}\cos t\sqrt{\frac{m^2\pi^2a^2}{l^2} - k^2}.\int_0^l f(\lambda)\sin\frac{m\pi\lambda}{l}\,d\lambda\right) \tag{12}$$

reduces to (11) when $t=0$ and therefore satisfies (4).

$$D_ty = -\frac{2}{le^{kt}}\sum_{m=1}^{m=\infty}\left(\sqrt{\frac{m^2\pi^2a^2}{l^2} - k^2}.\sin\frac{m\pi x}{l}\sin t\sqrt{\frac{m^2\pi^2a^2}{l^2} - k^2}.\int_0^l f(\lambda)\sin\frac{m\pi\lambda}{l}\,d\lambda\right)$$

$$-\frac{2k}{le^{kt}}\sum_{m=1}^{m=\infty}\left(\sin\frac{m\pi x}{l}\cos t\sqrt{\frac{m^2\pi^2a^2}{l^2} - k^2}.\int_0^l f(\lambda)\sin\frac{m\pi\lambda}{l}\,d\lambda\right). \tag{13}$$

When $t=0$ the first line of the second member of (13) vanishes but the second line reduces to

$$-\frac{2k}{l}\sum_{m=1}^{m=\infty}\left(\sin\frac{m\pi x}{l}\int_0^l f(\lambda)\sin\frac{m\pi\lambda}{l}\,d\lambda\right).$$

We must, then, introduce into (12) an additional term which shall equal zero when $t=0$ and whose derivative with respect to t shall cancel the term above when $t=0$.

This is easily seen to be

$$\frac{2k}{l}e^{-kt}\sum_{m=1}^{m=\infty}\frac{1}{\sqrt{\frac{m^2\pi^2a^2}{l^2}-k^2}}\sin\frac{m\pi x}{l}\sin t\sqrt{\frac{m^2\pi^2a^2}{l^2}-k^2}.\int_0^l f(\lambda)\sin\frac{m\pi\lambda}{l}\,d\lambda\,.$$

Hence our complete solution is

$$y=\frac{2}{l}e^{-kt}\sum_{m=1}^{m=\infty}\Bigg[\Bigg(\cos t\sqrt{\frac{m^2\pi^2a^2}{l^2}-k^2}$$

$$+\frac{k}{\sqrt{\frac{m^2\pi^2a^2}{l^2}-k^2}}\sin t\sqrt{\frac{m^2\pi^2a^2}{l^2}-k^2}\Bigg)\sin\frac{m\pi x}{l}\int_0^l f(\lambda)\sin\frac{m\pi\lambda}{l}\,d\lambda\Bigg]. \quad (14)$$

Here the fact that e^{-kt}, which decreases rapidly as t increases, is a factor of the whole second member shows that the amplitude of the vibration rapidly decreases.

Comparing this solution with that given in Art. 64 for the case where there is no resistance we see that the period of any given term

$$A\sin\frac{m\pi x}{l}\cos t\sqrt{\frac{m^2\pi^2a^2}{l^2}-k^2},$$

is greater than that of the corresponding term $A_1\sin\frac{m\pi x}{l}\cos\frac{m\pi at}{l}$ in Art. 64. In other words the effect of the resistance of the air is to flatten somewhat each component part of the note given by the string. More than this since the periods of the different terms of (14) are no longer exact submultiples of the period of the first term, the component notes are no longer in perfect harmony with the fundamental note of the string, and the ideal perfect harmony between the fundamental note and its harmonics is not quite realized in any actual case.

When k is very small, as in the case of a fine string, the departure from perfect harmony is very slight; but in the case of a coarse string or worse still of an elastic ribbon, where the resistance of the air is considerable, the unmusical character of the sound is very noticeable.

EXAMPLES.

1. Solve Ex. 1 Art. 64 allowing for the resistance of the air.
2. Solve Ex. 2 Art. 64 allowing for the resistance of the air;

$$y=\frac{2}{l}e^{-kt}\sum_{m=1}^{m=\infty}\Bigg(\frac{1}{\sqrt{\frac{m^2\pi^2a^2}{l^2}-k^2}}\sin\frac{m\pi x}{l}\sin t\sqrt{\frac{m^2\pi^2a^2}{l^2}-k^2}.\int_0^l F(\lambda)\sin\frac{m\pi\lambda}{l}\,d\lambda\Bigg).$$

3. Find a particular solution of (1) Art. 65 on the assumption that the form $y = T.X$, where T is a function of t alone and X a functio alone.

66. We pass on now to a couple of problems that require the modi and extension of Fourier's Theorem, the *cooling of a sphere in air*, a *vibration of a stretched rectangular membrane*, but as an introduction former we shall first consider the following very simple problem; to fi temperature of any point of a sphere whose initial temperature is any function of r the distance of the point from the centre, and whose sur kept at the constant temperature b.

Here we are to solve

$$D_t(ru) = a^2 D_r^2(ru),$$

see [V] Art. 1, subject to the conditions

$$u = f(r) \text{ when } t = 0$$

$$u = b \quad " \quad r = c$$

if c is the radius.

Let $v = ru$, then our equations become

$$D_t v = a^2 D_r^2 v$$

$$v = r f(r) \text{ when } t = 0$$

$$v = bc \quad " \quad r = c$$

$$v = 0 \quad " \quad r = 0.$$

Our problem is now precisely that of Art. 62 and we have as our solutio

$$ru = \frac{2}{c}\sum_{m=1}^{m=\infty}\left(e^{-\frac{m^2\pi^2a^2t}{c^2}}\sin\frac{m\pi r}{c}\int_0^c \lambda f(\lambda)\sin\frac{m\pi\lambda}{c}\,d\lambda\right)$$

$$+\,b\left[r+\frac{2c}{\pi}\sum_{m=1}^{m=\infty}\left(\frac{(-1)^m}{m}e^{-\frac{m^2\pi^2a^2t}{c^2}}\sin\frac{m\pi r}{c}\right)\right].$$

EXAMPLES.

1. If $f(r) = b$ (8) Art. 66 reduces to $u = b$ and there is no chan temperature.

2. If the initial temperature is constant and equal to β

$$u = b + \frac{2c}{\pi r}(\beta - b)\left[e^{-\frac{\pi^2a^2t}{c^2}}\sin\frac{\pi r}{c} - \frac{1}{2}e^{-\frac{4\pi^2a^2t}{c^2}}\sin\frac{2\pi r}{c}\right.$$

$$\left. + \frac{1}{3}e^{-\frac{9\pi^2a^2t}{c^2}}\sin\frac{3\pi r}{c} - \ldots\right].$$

3. An iron sphere 40 cm. in diameter is heated to the temperature 100° centigrade throughout; its surface is then kept at the constant temperature 0°. Find the temperature of a point 10 cm. from the centre, and find the temperature of the centre, 15 minutes after cooling has begun. Given $a^2 = 0.185$ in C.G.S. units. *Ans.*, 2°.1; 3°.3.

67. If instead of having the temperature of the surface of the sphere constant, the sphere is placed in air which is kept at the constant temperature zero, the problem is much more complicated. For in this case the surface temperature can no longer be simply expressed but is given by a new differential equation

$$D_r u + hu = 0 \quad \text{when} \quad r = c, \tag{1}$$

where h is an experimental constant depending upon what is called the surface conductivity of the sphere.

Our equations, then, are

$$D_t(ru) = a^2 D_r^2(ru) \tag{2}$$

$$u = f(r) \quad \text{when} \quad t = 0 \tag{3}$$

$$D_r u + hu = 0 \quad \text{when} \quad r = c. \tag{4}$$

As in Art. 66 let $v = ru$; then we have

$$D_t v = a^2 D_r^2 v \tag{5}$$

$$v = rf(r) \quad \text{when} \quad t = 0 \tag{6}$$

$$v = 0 \quad \text{"} \quad r = 0 \tag{7}$$

$$D_r v + \left(h - \frac{1}{c}\right) v = 0 \quad \text{when} \quad r = c. \tag{8}$$

$v = e^{-a^2\alpha^2 t}\cos \alpha r$ and $v = e^{-a^2\alpha^2 t}\sin \alpha r$ have already been found as particular solutions of (5) (see Art. 60).

$$v = e^{-a^2\alpha^2 t}\sin \alpha r \tag{9}$$

satisfies (7) for all values of α.

Substitute this value of v in (8) and we have

$$\alpha c \cos \alpha c + (hc - 1)\sin \alpha c = 0. \tag{10}$$

If α_k is a value of α which is a root of the transcendental equation (10)

$$v = e^{-a^2\alpha_k^2 t}\sin \alpha_k r \tag{11}$$

will satisfy (5), (7), and (8).

It remains to see whether out of terms of the form given in (11) we can build up a value of v which will satisfy (6).

When $t = 0$ the second member of (11) reduces to $\sin\alpha_k x$. If then we can express $xf(x)$ as a sum of terms of the form $b_k \sin\alpha_k x$ where α_k is a root of (10)

$$xu = \sum b_k e^{-a^2\alpha_k^2 t}\sin\alpha_k x \tag{12}$$

will satisfy all of the equations (5), (6), (7), and (8), and will be the required solution.

Here, then, we have a new problem analogous to that of developing in a Fourier's Series, but rather more complicated, namely, to develop any function of x in a series of the form $\sum a_m \sin\alpha_m x$ where α_m is a root of the equation (10); or if we call $\alpha c = \phi$ and $hc - 1 = p$, where $\alpha_m = \dfrac{\phi_m}{c}$, ϕ_m being a root of the equation

$$\phi\cos\phi + p\sin\phi = 0 \tag{13}$$

or more simply of

$$\phi + p\tan\phi = 0, \tag{14}$$

remembering that the series and the function must be equal for all values of x between zero and c.

If ϕ_m is a root of (14) $-\phi_m$ is also a root.

Since $\sin\dfrac{\phi_m}{c}x = -\sin\left(-\dfrac{\phi_m}{c}x\right)$ the terms of the required development which correspond to negative roots may be combined with those corresponding to positive roots, and therefore we need consider only positive roots.

$\phi = 0$ is a root of (14) but as $\sin 0 = 0$ there will be no corresponding term in the development.

If we construct the curve

$$y = -\frac{1}{p}x \tag{15}$$

and the curve

$$y = \tan x \tag{16}$$

the abscissas of their points of intersection are values of x which satisfy $\dfrac{x}{p} + \tan x = 0$, that is, are roots of equation (14). It is easy to see that there will always be an infinite number of real positive roots, one for each of the branches of the periodic curve $y = \tan x$ which lie to the right of the origin. The numerical values of these roots can be obtained by an easy computation. The construction suggested above shows that as m increases ϕ_m will rapidly approach the value $(2m-1)\dfrac{\pi}{2}$ if p is positive or if p is negative and numerically less than unity, and $(2m+1)\dfrac{\pi}{2}$ if p is negative and numerically greater than unity.

There exist, then, an infinite number of positive real roots of $\phi + p \tan \phi = 0$ and consequently of

$$ac \cos ac + (hc - 1) \sin ac = 0 .$$

68. The development called for in the last article can be obtained very easily from a simpler one which we shall now consider, namely, to develop $f(x)$ into a series of the form

$$f(x) = a_1 \sin \phi_1 x + a_2 \sin \phi_2 x + a_3 \sin \phi_3 x + \cdots \tag{1}$$

where $\phi_1, \phi_2, \phi_3 \cdots$ are roots of the equation

$$\phi \cos \phi + p \sin \phi = 0 , \tag{2}$$

the development to hold good for all values of x between $x = 0$ and $x = 1$.

Let us proceed as in Arts. 24 and 27. Call $\dfrac{1}{n+1} = \Delta x$ and form n equations by substituting for x in turn in the equation

$$f(x) = a_1 \sin \phi_1 x + a_2 \sin \phi_2 x + a_3 \sin \phi_3 x + \cdots + a_n \sin \phi_n x \tag{3}$$

the values $\Delta x, 2\Delta x, 3\Delta x, \cdots n\Delta x$; this being equivalent to making the values of the sum and the function coincide for the n values of x substituted.

To determine any coefficient a_m multiply the first equation by $\Delta x. \sin (\phi_m \Delta x)$, the second by $\Delta x. \sin (2\phi_m \Delta x)$, the third by $\Delta x. \sin (3\phi_m \Delta x)$, and so on, the nth equation by $\Delta x. \sin (n\phi_m \Delta x)$; add the equations and compute the limiting values of the terms of the resulting equation as n is indefinitely increased. This as in Art. 24 is seen to be equivalent to multiplying (3) by $\sin \phi_m x.dx$ and integrating between the limits $x = 0$ and $x = 1$.

The first member of the resulting equation is

$$\int_0^1 f(x) \sin \phi_m x.dx ;$$

The coefficient of a_k is

$$\int_0^1 \sin \phi_k x \sin \phi_m x.dx ,$$

and of a_m is

$$\int_0^1 \sin^2 \phi_m x.dx .$$

$$\int_0^1 \sin \phi_k x \sin \phi_m x . dx = \frac{1}{2}\int_0^1 [\cos (\phi_k - \phi_m)x - \cos (\phi_k + \phi_m)x] dx$$

$$= \frac{1}{2}\left[\frac{\sin (\phi_k - \phi_m)}{\phi_k - \phi_m} - \frac{\sin (\phi_k + \phi_m)}{\phi_k + \phi_m}\right]$$

$$= - \frac{\phi_k \cos \phi_k \sin \phi_m - \phi_m \sin \phi_k \cos \phi_m}{\phi_k^2 - \phi_m^2}.$$

But $$\phi_k \cos \phi_k + p \sin \phi_k = 0$$

and $$\phi_m \cos \phi_m + p \sin \phi_m = 0 \qquad \text{by (2).}$$

Hence the numerator of the second member of (4) is zero, and the co of a_k vanishes if k is not equal to m.

$$\int_0^1 \sin^2 \phi_m x . dx = \frac{1}{2\phi_m}[\phi_m - \sin \phi_m \cos \phi_m] = \frac{1}{2}\left[1 - \frac{\sin 2\phi_m}{2\phi_m}\right].$$

Therefore $$a_m = \frac{2}{1 - \dfrac{\sin 2\phi_m}{2\phi_m}}\int_0^1 f(x) \sin \phi_m x . dx.$$

The coefficient of the integral in (6) can be transformed as follows s to involve trigonometric functions.

$$\phi_m \cos \phi_m + p \sin \phi_m = 0, \qquad \text{by (2)}$$

$$\phi_m \cos^2 \phi_m + \frac{p}{2} \sin 2\phi_m = 0,$$

$$\frac{\sin 2\phi_m}{2\phi_m} = -\frac{\cos^2 \phi_m}{p}.$$

$$\phi_m^2 \cos^2 \phi_m = p^2 \sin^2 \phi_m,$$

$$(\phi_m^2 + p^2) \cos^2 \phi_m = p^2,$$

$$\frac{\cos^2 \phi_m}{p} = \frac{p}{\phi_m^2 + p^2}.$$

Hence by (7) and (8)

$$1 - \frac{\sin 2\phi_m}{2\phi_m} = \frac{\phi_m^2 + p(p+1)}{\phi_m^2 + p^2},$$

and $$a_m = \frac{2(\phi_m^2 + p^2)}{\phi_m^2 + p(p+1)}\int_0^1 f(a) \sin \phi_m a . da.$$

Therefore our required development is

$$f(x) = \sum_{m=1}^{m=\infty}\left(\frac{2(\phi_m^2 + p^2)}{\phi_m^2 + p(p+1)} \sin \phi_m x \int_0^1 f(a) \sin \phi_m a . da\right).$$

From (10) it easily follows that for values of x between 0 and c

$$f(x) = a_1 \sin \alpha_1 x + a_2 \sin \alpha_2 x + a_3 \sin \alpha_3 x + \cdots \tag{11}$$

where
$$a_m = \frac{2}{c} \cdot \frac{\alpha_m^2 c^2 + p^2}{\alpha_m^2 c^2 + p(p+1)} \int_0^c f(\lambda) \sin \alpha_m \lambda . d\lambda, \tag{12}$$

and α_m is a root of the equation

$$\alpha c \cos \alpha c + p \sin \alpha c = 0 . \tag{13}$$

It is to be observed that if p is infinite (13) reduces to $\sin \alpha c = 0$, α_m becomes $\frac{m\pi}{c}$ and (11) and (12) give our regulation Fourier sine series (v. Art. 31), and therefore the ordinary Fourier development in sine series is merely a special case of the problem just solved.

Moreover since the Fourier method of determining the coefficients of such a series requires that

$$\int_0^c \sin \alpha_m x \sin \alpha_n x . dx = 0 ,$$

that is that
$$\frac{\sin(\alpha_m - \alpha_n)c}{\alpha_m - \alpha_n} - \frac{\sin(\alpha_m + \alpha_n)c}{\alpha_m + \alpha_n} = 0$$

or reducing, that
$$\frac{\alpha_m c \cos \alpha_m c}{\sin \alpha_m c} = \frac{\alpha_n c \cos \alpha_n c}{\sin \alpha_n c} ,$$

or that α_m and α_n should be roots of the equation

$$\frac{\alpha c \cos \alpha c}{\sin \alpha c} = p$$

where p is some constant, it follows that we have obtained in (11) the most general sine development that can be obtained by Fourier's method.

EXAMPLES.

1. Show that the solution of the problem of Art. 67 is

$$ru = \sum_{m=1}^{m=\infty} b_m e^{-a^2 \alpha_m^2 t} \sin \alpha_m r ,$$

where
$$b_m = \frac{2}{c} \cdot \frac{\alpha_m^2 c^2 + (hc-1)^2}{\alpha_m^2 c^2 + hc(hc-1)} \int_0^c \lambda f(\lambda) \sin \alpha_m \lambda . d\lambda$$

and α_m is a root of

$$\alpha c \cos \alpha c + (hc - 1) \sin \alpha c = 0 .$$

2. If the initial temperature of the sphere is constant and equal to β

$$ru = \sum_{m=1}^{m=\infty} b_m e^{-a^2\alpha_m^2 t}\sin\alpha_m r$$

where $$b_m = 2\beta h\cdot\frac{\alpha_m^2c^2+(hc-1)^2}{\alpha_m^2c^2+hc(hc-1)}\cdot\frac{\sin\alpha_m c}{\alpha_m^2}$$

$$=\frac{2\beta hc}{\alpha_m}\cdot\frac{[\alpha_m^2c^2+(hc-1)^2]^{\frac{1}{2}}}{\alpha_m^2c^2+hc(hc-1)}.$$

3. If the temperature of the air is a constant γ instead of zero the surface equation of condition is

$$D_r u + h(u-\gamma) = 0 \quad \text{when} \quad r = c.$$

The substitution of $u_1 = u - \gamma$, however, brings the problem under Ex. 1 and we get

$$r(u-\gamma) = \sum_{m=1}^{m=\infty} b_m e^{-a^2\alpha_m^2 t}\sin\alpha_m r$$

where $$b_m = \frac{2}{c}\cdot\frac{\alpha_m^2c^2+(hc-1)^2}{\alpha_m^2c^2+hc(hc-1)}\int_0^c \lambda[f(\lambda)-\gamma]\sin\alpha_m\lambda.d\lambda.$$

4. An iron sphere 40 cm. in diameter is heated to the temperature 100° centigrade throughout; it is then allowed to cool in air which is kept at the constant temperature 0°. Find the temperature at the centre; at a point 10 cm. from the centre; and at the surface; 15 minutes after cooling has begun. Given $a^2 = 0.185$ and $h = \frac{1}{800}$ in C.G.S. units. (v. Ex. 3, Art. 66.)

Ans., 97°.67; 97°.36; 96°.46.

5. Show that if in the slab considered in Art. 60 one face is exposed to air at the temperature zero, so that we have $D_t u = a^2 D_x^2 u$, $u = 0$ when $x = 0$, $u = f(x)$ when $t = 0$, and $D_x u + hu = 0$ when $x = c$, then

$$u = \sum_{m=1}^{m=\infty} a_m e^{-a^2\alpha_m^2 t}\sin\alpha_m x$$

where $$a_m = 2\,\frac{\alpha_m^2+h^2}{\alpha_m^2 c + h(hc+1)}\int_0^c f(\lambda)\sin\alpha_m\lambda.d\lambda,$$

α_m being a root of $\alpha c\cos\alpha c + hc\sin\alpha c = 0$.

6. If in the problem of Art. 57 heat escapes from one side of the plate into air at the temperature zero so that we have $D_x^2 u + D_y^2 u = 0$, $u = 0$ when $x = 0$, $u = f(x)$ when $y = 0$, and $D_x u + hu = 0$ when $x = a$, then

$$u = \sum_{m=1}^{m=\infty} a_m e^{-\alpha_m y} \sin \alpha_m x$$

where $$a_m = 2\frac{\alpha_m^2 + h^2}{\alpha_m^2 a + h(ha+1)}\int_0^a f(\lambda) \sin \alpha_m \lambda . d\lambda ,$$

α_m being a root of $\alpha a \cos \alpha a + ha \sin \alpha a = 0$.

7. If in the problem of Art. 59 there is leakage at one side of the sheet so that we have $D_x^2 V + D_y^2 V = 0$, $V = 0$ when $x = 0$, $V = 0$ when $y = b$, $V = f(x)$ when $y = 0$, and $D_x V + hV = 0$ when $x = a$, then

$$V = \sum_{m=1}^{m=\infty} a_m \frac{\sinh \alpha_m (b - y)}{\sinh \alpha_m b} \sin \alpha_m x,$$

where a_m has the value given in Ex. 6.

69. If we have an infinite solid with one plane face which is exposed to air at the temperatures $U = F(t)$ and heat can flow only at right angles to this face, we can solve the problem readily for the case where the initial temperatures are zero. We have

$$D_t u = a^2 D_x^2 u$$

subject to the conditions

$$u = 0 \quad \text{when} \quad t = 0$$

and $$D_x u + h(U - u) = 0 \quad \text{when} \quad x = 0.$$

Let $$v = u - \frac{1}{h} D_x u. \tag{1}$$

Then v will satisfy the equation

$$D_t v = a^2 D_x^2 v,$$

and we shall also have $v = U$ when $x = 0$.

Since $U = F(t)$ $$v = \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\beta^2} F\left(t - \frac{x^2}{4a^2\beta^2}\right) d\beta \tag{2}$$

by Art. **51 (10)**.

$$D_x u - hu = -hv \quad \text{by (1).}$$

Hence $$ue^{-hx} = -h\int e^{-hx} v dx + C;$$

v. Int. Cal. § 4, page 314.

Determining C by the fact that $ue^{-hx} = 0$ when $x = \infty$ we have

$$u = he^{hx}\int_x^\infty e^{-hx} v\,dx\,. \tag{3}$$

Substituting the value of v from (2) we have

$$u = \frac{2he^{hx}}{\sqrt{\pi}}\int_x^\infty e^{-hx}dx\int_{\frac{x}{2a\sqrt{t}}}^\infty e^{-\beta^2}F\left(t-\frac{x^2}{4a^2\beta^2}\right)d\beta\,, \tag{4}$$

as our required solution.

For an extension of this method to the flow of heat in two and three dimensions and for the interpretation of the results by the aid of the theory of *Images*, see E. W. Hobson, Proc. Lond. Math. Soc., Vol. XIX.

EXAMPLES.

1. If the temperature of the air is a periodic function of the time, say $\rho_m \sin(mat+\lambda_m)$ and we care only for the limiting value of u as t increases, show that this value is

$$\frac{h\rho_m e^{-\frac{x}{a}\sqrt{\frac{ma}{2}}}}{\left(h+\frac{1}{a}\sqrt{\frac{ma}{2}}\right)^2+\frac{ma}{2a^2}}\left[\left(h+\frac{1}{a}\sqrt{\frac{ma}{2}}\right)\sin\left(mat-\frac{x}{a}\sqrt{\frac{ma}{2}}+\lambda_m\right)\right.$$
$$\left.-\frac{1}{a}\sqrt{\frac{ma}{2}}\cos\left(mat-\frac{x}{a}\sqrt{\frac{ma}{2}}+\lambda_m\right)\right].$$

v. Art. 52 and Art. 51 Ex. 4.

Note that $$\int e^{ax}\sin bx.dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

and $$\int e^{ax}\cos bx.dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}.$$

v. Int. Cal. Table of Int. (235) and (236).

2. If $D_x^2V + D_y^2V = 0$, $V=0$ when $y=0$ and $D_xV + h[F(y) - V] = 0$ when $x=0$ show that

$$V = \frac{he^{hx}}{\pi}\int_x^\infty e^{-hx}dx\int_0^\infty F(\lambda)d\lambda\left[\frac{x}{x^2+(\lambda-y)^2} - \frac{x}{x^2+(\lambda+y)^2}\right];$$

v. Art. 47 Ex. 1.

70. The solution for an instantaneous heat source of strength Q at the point $x=\lambda$ if heat escapes at the origin into air at the temperature zero, so that $D_xu - hu = 0$ when $x=0$, can be obtained by the aid of Art. 53.

Let $u = u_1 + u_2$ where u_1 is the temperature that would be due to the given source if we had no boundary at the origin, so that

$$u_1 = \frac{Q}{2a\sqrt{\pi t}} e^{-\frac{(\lambda - x)^2}{4a^2 t}}. \qquad \text{[Art. 53 (2)]}$$

$$D_x u - hu = D_x u_1 - hu_1 + D_x u_2 - hu_2 = 0 \quad \text{when} \quad x = 0.$$

Therefore
$$D_x u_2 - hu_2 = -(D_x u_1 - hu_1) \tag{1}$$
when $x = 0$.

But
$$-(D_x u_1 - hu_1) = -\frac{Q}{2a\sqrt{\pi t}}\left(\frac{\lambda - x}{2a^2 t} - h\right) e^{-\frac{(\lambda - x)^2}{4a^2 t}}$$

$$= -\frac{Q}{2a\sqrt{\pi t}}\left(\frac{\lambda}{2a^2 t} - h\right) e^{-\frac{\lambda^2}{4a^2 t}}$$

when $x = 0$.

This is easily seen to be the value to which

$$-\frac{Q}{2a\sqrt{\pi t}}\left(\frac{\lambda + x}{2a^2 t} - h\right) e^{-\frac{(\lambda + x)^2}{4a^2 t}}$$

reduces when $x = 0$, and this last expression is

$$(D_x + h)\frac{Q}{2a\sqrt{\pi t}} e^{-\frac{(\lambda + x)^2}{4a^2 t}}$$

and therefore satisfies the equation

$$D_t u = a^2 D_x^2 u; \tag{2}$$

since $\dfrac{Q}{2a\sqrt{\pi t}} e^{-\frac{(\lambda + x)^2}{4a^2 t}}$ is the temperature due to a source at $x = -\lambda$.

If, then, we determine u_2 from the condition that

$$D_x u_2 - hu_2 = -\frac{Q}{2a\sqrt{\pi t}}\left(\frac{\lambda + x}{2a^2 t} - h\right) e^{-\frac{(\lambda + x)^2}{4a^2 t}} \tag{3}$$

taking care not to introduce any arbitrary constant or arbitrary function of t in our integration, u_2 will satisfy equation (2) and condition (1).

Integrating (3) [v. Int. Cal. § 4, page 314] and determining the constants of integration suitably we get

$$u_2 = \frac{Q}{2a\sqrt{\pi t}}\left[e^{-\frac{(\lambda + x)^2}{4a^2 t}} - 2he^{hx}\int_x^\infty e^{-hx - \frac{(\lambda + x)^2}{4a^2 t}} dx\right]. \tag{4}$$

Therefore the solution of our problem is

$$u = \frac{Q}{2a\sqrt{\pi t}}\left[e^{-\frac{(\lambda - x)^2}{4a^2 t}} + e^{-\frac{(\lambda + x)^2}{4a^2 t}} - 2he^{hx}\int_x^\infty e^{-hx - \frac{(\lambda + x)^2}{4a^2 t}} dx\right]. \tag{5}$$

If we replace Q by $f(\lambda)d\lambda$ and integrate from 0 to ∞ we get as the solution for the case where $u = f(x)$ when $t = 0$ and $x > 0$, and $D_x u - hu = 0$ when $x = 0$

$$u = \frac{1}{2a\sqrt{\pi t}}\int_0^\infty f(\lambda)d\lambda\left[e^{-\frac{(\lambda-x)^2}{4a^2t}} + e^{-\frac{(\lambda+x)^2}{4a^2t}} - 2he^{hx}\int_x^\infty e^{-hx-\frac{(\lambda+x)^2}{4a^2t}}dx\right]. \qquad (6)$$

For an interpretation of this result by the theory of Images and the extension of the method to the conduction of heat in n dimensions see G. H. Bryan, Proc. Lond. Math. Soc., Vol. XXII.

EXAMPLE.

Show that if $u = f(x)$ when $t = 0$ and $D_x u + h[F(t) - u] = 0$ when $x = 0$ we must take u equal to the sum of the second members of (6) Art. 70 and of (4) Art. 69.

71. As another problem requiring a slight extension of Fourier's Theorem let us consider the vibration of a rectangular stretched elastic membrane fastened at the edges, that is of a rectangular drumhead.

If two of the sides are taken as axes and the plane of equilibrium of the membrane as the plane of XY the equation for the motion of the membrane is

$$D_t^2 z = c^2(D_x^2 z + D_y^2 z) \qquad (1)$$

see [x] Art. 1.

Let the membrane be distorted at the start into some given form $z = f(x, y)$ and then allowed to swing. Our equations of conditions are then

$$z = 0 \quad \text{when} \quad x = 0 \qquad (2)$$

$$z = 0 \quad \text{"} \quad x = a \qquad (3)$$

$$z = 0 \quad \text{"} \quad y = 0 \qquad (4)$$

$$z = 0 \quad \text{"} \quad y = b \qquad (5)$$

$$z = f(x, y) \quad \text{"} \quad t = 0 \qquad (6)$$

$$D_t z = 0 \quad \text{"} \quad t = 0. \qquad (7)$$

We can get a particular solution of (1) by our usual device. Assume

$$z = e^{ax + \beta y + \gamma t}$$

and substitute in (1). We get $\gamma^2 = c^2(a^2 + \beta^2)$ as the only relation that need hold between a, β, and γ, in order that $z = e^{ax+\beta y+\gamma t}$ may be a solution. This gives

$$\gamma = \pm c\sqrt{a^2 + \beta^2}.$$

Therefore $$z = e^{ax + \beta y \pm ct\sqrt{a^2+\beta^2}}$$

is a solution of (1) no matter what values are given to a and β.

Replace α and β by αi and βi and we have

$$z = e^{(\alpha x + \beta y \pm ct\sqrt{\alpha^2+\beta^2})i}$$

as a solution, and from this we get

$$z = \sin\left(\alpha x + \beta y \pm ct\sqrt{\alpha^2+\beta^2}\right) \tag{8}$$

and

$$z = \cos\left(\alpha x + \beta y \pm ct\sqrt{\alpha^2+\beta^2}\right) \tag{9}$$

as particular solutions of (1), α and β being unrestricted.

From (8) and (9) we can get solutions of the following forms

$$\left.\begin{aligned}
z &= \sin \alpha x \sin \beta y \sin ct\sqrt{\alpha^2+\beta^2}\\
z &= \sin \alpha x \sin \beta y \cos ct\sqrt{\alpha^2+\beta^2}\\
z &= \sin \alpha x \cos \beta y \sin ct\sqrt{\alpha^2+\beta^2}\\
z &= \sin \alpha x \cos \beta y \cos ct\sqrt{\alpha^2+\beta^2}\\
z &= \cos \alpha x \sin \beta y \sin ct\sqrt{\alpha^2+\beta^2}\\
z &= \cos \alpha x \sin \beta y \cos ct\sqrt{\alpha^2+\beta^2}\\
z &= \cos \alpha x \cos \beta y \sin ct\sqrt{\alpha^2+\beta^2}\\
z &= \cos \alpha x \cos \beta y \cos ct\sqrt{\alpha^2+\beta^2},
\end{aligned}\right\} \tag{10}$$

each of which will satisfy equation (1). The second of these will satisfy also (2), (4) and (7) whatever values be taken for α and β. It will satisfy (3) and (5) if α and β are equal $\dfrac{m\pi}{a}$ and $\dfrac{n\pi}{b}$ respectively.

If, then, we can so combine terms of the form

$$\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos c\pi t\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$$

as to satisfy (6) our problem will be completely solved.

This can be done if we can express $f(x, y)$ as a sum of terms of the form $A\sin\dfrac{m\pi x}{a}\sin\dfrac{n\pi y}{b}$, the sum and the function being equal when x lies between 0 and a and y between 0 and b.

$f(x, y)$ can be expressed in terms of $\sin\dfrac{m\pi x}{a}$ by Fourier's Theorem if we regard y as constant. We have

$$f(x, y) = \sum_{m=1}^{m=\infty} a_m \sin\frac{m\pi x}{a} \tag{11}$$

where $$a_m = \frac{2}{a}\int_0^a f(\lambda, y)\sin\frac{m\pi\lambda}{a}d\lambda. \tag{12}$$

$f(\lambda, y)$ in (12) is a function of y and may be developed by Fourier's Theorem.

We have $$f(\lambda, y) = \sum_{n=1}^{n=\infty} b_n \sin\frac{n\pi y}{b} \tag{13}$$

where $$b_n = \frac{2}{b}\int_0^b f(\lambda, \mu)\sin\frac{n\pi\mu}{b}d\mu. \tag{14}$$

Substituting for $f(\lambda, y)$ in (12) the value just obtained we have

$$a_m = \frac{2}{a}\frac{2}{b}\sum_{n=1}^{n=\infty}\left(\int_0^a d\lambda \int_0^b f(\lambda, \mu)\sin\frac{m\pi\lambda}{a}\sin\frac{n\pi\mu}{b}d\mu\right)\sin\frac{n\pi y}{b}$$

and

$$f(x, y) = \frac{4}{ab}\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\left(\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\int_0^a d\lambda\int_0^b f(\lambda, \mu)\sin\frac{m\pi\lambda}{a}\sin\frac{n\pi\mu}{b}d\mu\right). \tag{15}$$

Hence $$z = \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\left(A_{m,n}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos c\pi t\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\right), \tag{16}$$

where $$A_{m,n} = \frac{4}{ab}\int_0^a d\lambda\int_0^b f(\lambda, \mu)\sin\frac{m\pi\lambda}{a}\sin\frac{n\pi\mu}{b}d\mu. \tag{17}$$

is our required solution.

EXAMPLES.

1. Show that if the membrane starts from its position of equilibrium but with a given initial velocity impressed upon each point so that $z = 0$ when $t = 0$ and $D_t z = F(x, y)$ when $t = 0$ the solution is

$$z = \frac{1}{c\pi}\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\left(A_{m,n}\frac{1}{\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin c\pi t\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\right)$$

where $$A_{m,n} = \frac{4}{ab}\int_0^a d\lambda\int_0^b F(\lambda, \mu)\sin\frac{m\pi\lambda}{a}\sin\frac{n\pi\mu}{b}d\mu.$$

2. If there is both initial distortion and initial velocity

$$z = \frac{4}{ab}\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left[A_{m,n}\cos c\pi t\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} + B_{m,n}\sin c\pi t\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\right]$$

where
$$A_{m,n} = \int_0^a d\lambda \int_0^b f(\lambda,\mu)\sin\frac{m\pi\lambda}{a}\sin\frac{n\pi\mu}{b}\,d\mu,$$

and
$$B_{m,n} = \frac{1}{c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}\int_0^a d\lambda\int_0^b F(\lambda,\mu)\sin\frac{m\pi\lambda}{a}\sin\frac{n\pi\mu}{b}\,d\mu.$$

3. Obtain a particular solution of (1) Art. 71 by assuming $z = T.X.Y.$ where T is a function of t alone, X of x alone, and Y of y alone.

72. A number of interesting conclusions can be drawn from the results of Art. 71 and Exs. 1 and 2.

(*a*) No one of the three values of z is in general a periodic function of t, and consequently a vibrating rectangular membrane will not in general give a musical note.

(*b*) A stretched rectangular membrane can be made to give a musical note by starting the vibration properly. For if the initial circumstances are such that the solution reduces to a single term, as will be the case if the initial distortion in the problem of Art. 71 be such that $f(x,y) = A_{m,n}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$, or the initial velocity in Ex. 1 be such that $F(x,y) = B_{m,n}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$, or the initial distortion and initial velocity in Ex. 2 be the values just given, then the vibration will be periodic and will have the period

$$T = \frac{2}{c\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}. \tag{1}$$

Since T is a function of m and n and m and n are any whole numbers, the same membrane is capable of giving a great variety of musical notes of different pitches. If m and n are both unity we get the lowest note the membrane can give, which is called its fundamental note. Its period

$$T_1 = \frac{2}{c\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} = \frac{2ab}{c\sqrt{a^2+b^2}}. \tag{2}$$

If m and n are both equal to k we get

$$T_k = \frac{2ab}{kc\sqrt{a^2+b^2}}; \tag{3}$$

therefore the membrane can be made to give any harmonic of its fundam
note.

More than this, since as we have seen

$$T_{m,n} = \frac{2}{c\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}}$$

is the period of any note the membrane can give, and since if m and
replaced by mk and nk we get

$$T_{mk,nk} = \frac{2}{ck\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}}$$

the membrane can sound all the harmonics of any note which it can giv

(*c*) In the case considered above, where the solution reduces to the
term

$$z = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left[A_{m,n}\cos c\pi t\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} + B_{m,n}\sin c\pi t\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\right.$$

if $x = \frac{a}{m}$, or $\frac{2a}{m}$, or $\frac{3a}{m}\cdots$ or $\frac{(m-1)a}{m}$, $z = 0$ for all values of

the lines $x = \frac{a}{m}$, $x = \frac{2a}{m}$, $\cdots x = \frac{(m-1)a}{m}$ remain at rest during the
vibration and are nodes. The same thing is true of the lines

$$y = \frac{b}{n},\ y = \frac{2b}{n},\ y = \frac{3b}{n},\ \cdots y = \frac{(n-1)b}{n}.$$

73. If the membrane is square it may have much more complicated
than if the length and breadth are unequal, as in this case the period
term of the general solution reduces to

$$T = \frac{2a}{c\sqrt{m^2+n^2}}$$

and there will in general be two terms having the same period, and a n
note of the pitch corresponding to that period may be produced by init
cumstances that bring in both terms. Thus

$$z = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\left[A_{m,n}\cos\frac{c\pi t}{a}\sqrt{m^2+n^2} + B_{m,n}\sin\frac{c\pi t}{a}\sqrt{m^2+n^2}\right.$$

$$+ \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}\left[A_{n,m}\cos\frac{c\pi t}{a}\sqrt{m^2+n^2} + B_{n,m}\sin\frac{c\pi t}{a}\sqrt{m^2+n^2}\right.$$

is a form of vibration that will give a musical note. Let us write this

$$z = \cos\frac{c\pi t}{a}\sqrt{m^2+n^2}\left[A\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} + B\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}\right]$$

$$+\sin\frac{c\pi t}{a}\sqrt{m^2+n^2}\left[C\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} + D\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}\right] \qquad (2)$$

and in studying the forms of musical vibration of which the membrane is capable we may take A, B, C, and D at pleasure. Consider the simple case where $A = C$ and $B = D$; then (2) reduces to

$$z = \left(A\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} + B\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}\right)\left(\cos\frac{c\pi t}{a}\sqrt{m^2+n^2}\right.$$

$$\left.+\sin\frac{c\pi t}{a}\sqrt{m^2+n^2}\right). \qquad (3)$$

Values of x and y that will reduce the first parenthesis in (3) to zero will correspond to points of the membrane remaining motionless during the vibration.

Let us consider a few cases at length.

(a) If $m = 1$ and $n = 1$, the first parenthesis in (3) becomes

$$(A + B)\sin\frac{\pi x}{a}\sin\frac{\pi y}{a},$$

which is equal to zero only when $x = 0$ or $y = 0$, or $x = a$ or $y = a$, that is, for the four edges of the membrane. If, then, the membrane is sounding its fundamental note it has no nodes.

(b) If $m = 1$ and $n = 2$, we have

$$A\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} + B\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a} = 0$$

to give the nodes.

Let $B = 0$, then $\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} = 0$, which is satisfied by $y = \frac{a}{2}$; and in addition to the edges the line $y = \frac{a}{2}$ is at rest and is a node.

If $A = 0$ $x = \frac{a}{2}$ is a node.

If $A = B$

$$\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} + \sin\frac{2\pi x}{a}\sin\frac{\pi y}{a} = 0$$

$$2\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\cos\frac{\pi y}{a} + 2\sin\frac{\pi x}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{a} = 0$$

$$\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\left(\cos\frac{\pi y}{a} + \cos\frac{\pi x}{a}\right) = 0.$$

The first factor gives the four edges of the membrane. The second written equal to zero gives

$$\cos\frac{\pi y}{a} = -\cos\frac{\pi x}{a} = \cos\left(\pi - \frac{\pi x}{a}\right)$$

$$\frac{\pi y}{a} = \pi - \frac{\pi x}{a}$$

$$x + y = a,$$

which is a diagonal of the square.

If $B = -A$

$$\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} - \sin\frac{2\pi x}{a}\sin\frac{\pi y}{a} = 0$$

$$\cos\frac{\pi y}{a} = \cos\frac{\pi x}{a}$$

$$x - y = 0,$$

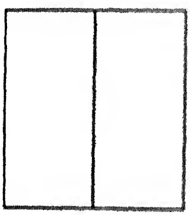
which is the other diagonal of the square.

Other relations between A and B will give Trigonometric curves of the form

$$\cos\frac{\pi y}{a} = -\frac{B}{A}\cos\frac{\pi x}{a}$$

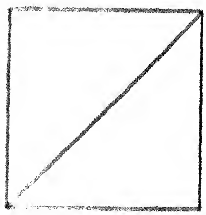
which are easily constructed and which obviously all agree in passing through the middle point of the square.

We give the figures for a few of the cases.

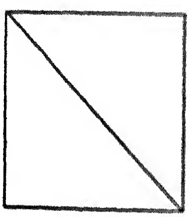


$A = 0$

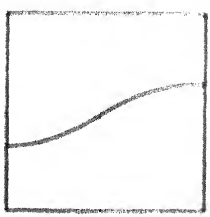
$B = 0$



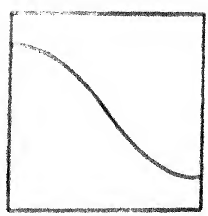
$A = -B$



$A = B$

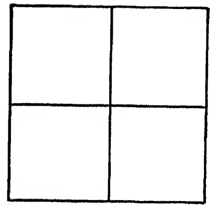


$A = 2B$



$A = \frac{1}{2}B$

(*c*) If $m=n=2$ we have



$$(A+B)\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a}=0$$

to give the nodes, which are merely the lines

$$x=\frac{a}{2}, \quad \text{and} \quad y=\frac{a}{2}.$$

This form gives the octave of the fundamental note.

(*d*) If $m=1$ and $n=3$ we have

$$A\sin\frac{\pi x}{a}\sin\frac{3\pi y}{a}+B\sin\frac{3\pi x}{a}\sin\frac{\pi y}{a}=0$$

to give the nodes.

If $A=0$ we get $\qquad x=\frac{a}{3}$ and $x=\frac{2a}{3}$ (1)

If $B=0$ we get $\qquad y=\frac{a}{3}$ and $y=\frac{2a}{3}$. (2)

If $A=-B$ we get

$$\sin\frac{\pi x}{a}\sin\frac{3\pi y}{a}-\sin\frac{3\pi x}{a}\sin\frac{\pi y}{a}=0$$

$$\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\left[4\cos^2\frac{\pi y}{a}-1-4\cos^2\frac{\pi x}{a}+1\right]=0$$

$$\cos^2\frac{\pi y}{a}-\cos^2\frac{\pi x}{a}=0$$

$$\left(\cos\frac{\pi y}{a}-\cos\frac{\pi x}{a}\right)\left(\cos\frac{\pi y}{a}+\cos\frac{\pi x}{a}\right)=0$$

or $\qquad x-y=0$ and $x+y=a$. (3)

If $A=B$ we get $\qquad \cos^2\frac{\pi y}{a}+\cos^2\frac{\pi x}{a}=\frac{1}{2}$

or $\qquad \cos\frac{2\pi y}{a}+\cos\frac{2\pi x}{a}=-1,$ (4)

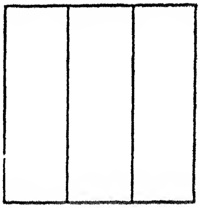
a Trigonometric curve easily constructed.

For other relations between A and B we get more complicated Trigonometric curves coming under the general form

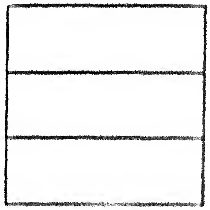
$$A\cos\frac{2\pi y}{a}+B\cos\frac{2\pi x}{a}=-\frac{A+B}{2} \qquad (5)$$

which all agree in containing the points

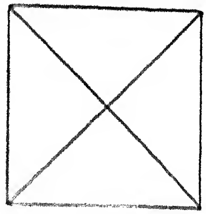
$$\left(\frac{a}{3}, \frac{a}{3}\right), \left(\frac{a}{3}, \frac{2a}{3}\right), \left(\frac{2a}{3}, \frac{a}{3}\right), \text{ and } \left(\frac{2a}{3}, \frac{2a}{3}\right).$$



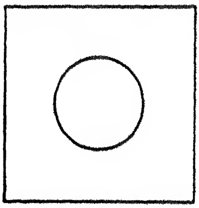
$A = 0$



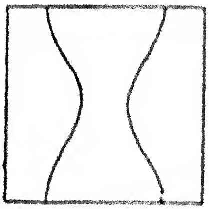
$B = 0$



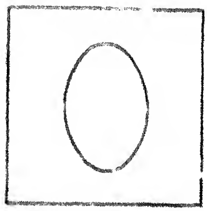
$A = -B$



$A = B$



$B = -2A$



$B = 2A$

MISCELLANEOUS PROBLEMS.

I. *Logarithmic Potential. Polar Coördinates.*

1. Show that $D_x^2 V + D_y^2 V = 0$ becomes

$$D_r^2 V + \frac{1}{r} D_r V + \frac{1}{r^2} D_\phi^2 V = 0$$

we transform to Polar Coördinates.

2. If in
$$D_r^2 V + \frac{1}{r} D_r V + \frac{1}{r^2} D_\phi^2 V = 0 \qquad (1)$$

let $V = R.\Phi$ we get

$$\left.\begin{aligned}\Phi &= A\cos a\phi + B\sin a\phi\\ R &= A_1 r^a + B_1 r^{-a}\end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned}\Phi &= Ae^{a\phi} + Be^{-a\phi}\\ R &= A_1\cos(a\log r) + B_1\sin(a\log r);\end{aligned}\right\}$$

ence

$V = r^a \cos a\phi$	$V = e^{a\phi}\cos(a\log r)$	$V = \cosh a\phi \cos(a\log r)$
$V = r^a \sin a\phi$	$V = e^{a\phi}\sin(a\log r)$	$V = \cosh a\phi \sin(a\log r)$
$V = \frac{1}{r^a}\cos a\phi$	$V = e^{-a\phi}\cos(a\log r)$	$V = \sinh a\phi \cos(a\log r)$
$V = \frac{1}{r^a}\sin a\phi$	$V = e^{-a\phi}\sin(a\log r)$	$V = \sinh a\phi \sin(a\log r)$

e particular solutions of (1).

3. Show that if V satisfies (1) Ex. 2 and $V = f(\phi)$ when $r = a$

$$V = \frac{1}{2}b_0 + \sum_{m=1}^{m=\infty}\left(\frac{r}{a}\right)^m (b_m\cos m\phi + a_m \sin m\phi) \quad \text{for} \quad r < a$$

1
$$V = \frac{1}{2}b_0 + \sum_{m=1}^{m=\infty}\left(\frac{a}{r}\right)^m (b_m\cos m\phi + a_m \sin m\phi) \quad \text{for} \quad r > a,$$

ere
$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\cos m\phi.d\phi \quad \text{and} \quad a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\sin m\phi.d\phi$$

4. Show that if V satisfies (1) Ex. 2 and $V=f(r)$ when $\phi=0$ and $r>0$

$$V=\frac{1}{\pi}\int_{-\infty}^{\infty}f(e^{\lambda})d\lambda\int_0^{\infty}\frac{\cosh a(\pi-\phi)}{\cosh a\pi}\cos a(\lambda-\log r).da$$

$$=\frac{1}{\pi}\sin\frac{\phi}{2}\int_{-\infty}^{\infty}f(e^{\lambda})\frac{\cosh\frac{1}{2}(\lambda-\log r)}{\cosh(\lambda-\log r)-\cos\phi}d\lambda.$$

5. If $V=1$ when $\phi=0$ and $0<r<1$, and $V=0$ when $\phi=0$ and $r>1$

$$V=\frac{1}{\pi}\left\{\frac{\pi}{2}-\tan^{-1}\left[\frac{\sinh\frac{\log r}{2}}{\sin\frac{\phi}{2}}\right]\right\}=\frac{1}{\pi}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{r-1}{2\sqrt{r}.\sin\frac{\phi}{2}}\right)\right].$$

6. If $V=f(r)$ when $\phi=0$ and $V=0$ when $\phi=\beta$

$$V=\frac{1}{\pi}\int_{-\infty}^{\infty}f(e^{\lambda})d\lambda\int_0^{\infty}\frac{\sinh(\beta-\phi)a}{\sinh\beta a}\cos a(\lambda-\log r).da$$

$$=\frac{1}{2\beta}\sin\frac{\pi\phi}{\beta}\int_{-\infty}^{\infty}\frac{f(e^{\lambda})d\lambda}{\cosh\frac{\pi}{\beta}(\lambda-\log r)-\cos\frac{\pi}{\beta}\phi},$$

if $0<\phi<\beta$.

7. If $V=0$ when $\phi=0$ and $V=F(r)$ when $\phi=\beta$

$$V=\frac{1}{\pi}\int_{-\infty}^{\infty}F(e^{\lambda})d\lambda\int_0^{\infty}\frac{\sinh\phi a}{\sinh\beta a}\cos a(\lambda-\log r).da$$

$$=\frac{1}{2\beta}\sin\frac{\pi\phi}{\beta}\int_{-\infty}^{\infty}\frac{F(e^{\lambda})d\lambda}{\cosh\frac{\pi}{\beta}(\lambda-\log r)+\cos\frac{\pi}{\beta}\phi}.$$

8. If $V=\chi(r)$ when $\phi=0$ and $r<a$, $V=0$ when $\phi=\beta$, and $V=0$ when $r=a$

$$V=\frac{1}{2\beta}\sin\frac{\pi\phi}{\beta}\int_{-\infty}^{0}\chi(ae^{\lambda})\left[\frac{d\lambda}{\cosh\frac{\pi}{\beta}\left(\lambda-\log\frac{r}{a}\right)-\cos\frac{\pi\phi}{\beta}}\right.$$

$$\left.-\frac{d\lambda}{\cosh\frac{\pi}{\beta}\left(\lambda+\log\frac{r}{a}\right)-\cos\frac{\pi\phi}{\beta}}\right].$$

9. If $V=0$ when $r=1$, $V=1$ when $\phi=0$, $V=0$ when $\phi=\frac{\pi}{2}$

$$V=\frac{2}{\pi}\tan^{-1}\left[\frac{1-r^2}{1+r^2}\operatorname{ctn}\phi\right].$$

10. If $V=0$ when $r=1$, $V=1$ when $\phi=0$, $V=1$ when $\phi=\frac{\pi}{2}$

$$V=\frac{2}{\pi}\tan^{-1}\left[\frac{1-r^4}{2r^2\sin 2\phi}\right].$$

11. If $V=f(\phi)$ when $r=a$, $V=0$ when $\phi=0$, and $V=0$ when $\phi=\beta$

$$V=\sum_{m=1}^{m=\infty}a_m\left(\frac{r}{a}\right)^{\frac{m\pi}{\beta}}\sin\frac{m\pi\phi}{\beta}\quad\text{if}\quad r<a$$

$$V=\sum_{m=1}^{m=\infty}a_m\left(\frac{a}{r}\right)^{\frac{m\pi}{\beta}}\sin\frac{m\pi\phi}{\beta}\quad\text{if}\quad r>a$$

where
$$a_m=\frac{2}{\beta}\int_0^\beta f(\phi)\sin\frac{m\pi\phi}{\beta}\,d\phi\quad\text{and}\quad 0<\phi<\beta.$$

12. If $V=f(\phi)$ when $r=a$, $V=0$ when $r=b$, $V=0$ when $\phi=0$, and $V=0$ when $\phi=\beta$, then if $a<r<b$ and $0<\phi<\beta$

$$V=\sum_{m=1}^{m=\infty}\left\{\frac{a^{\frac{m\pi}{\beta}}b^{\frac{m\pi}{\beta}}}{a^{\frac{2m\pi}{\beta}}-b^{\frac{2m\pi}{\beta}}}\left[\left(\frac{r}{b}\right)^{\frac{m\pi}{\beta}}-\left(\frac{b}{r}\right)^{\frac{m\pi}{\beta}}\right]a_m\sin\frac{m\pi\phi}{\beta}\right\}$$

where
$$a_m=\frac{2}{\beta}\int_0^\beta f(\phi)\sin\frac{m\pi\phi}{\beta}\,d\phi.$$

13. If $V=F(\phi)$ when $r=b$, $V=0$ when $r=a$, $V=0$ when $\phi=0$, and $V=0$ when $\phi=\beta$, then if $a<r<b$ and $0<\phi<\beta$

$$V=\sum_{m=1}^{m=\infty}\left\{\frac{a^{\frac{m\pi}{\beta}}b^{\frac{m\pi}{\beta}}}{b^{\frac{2m\pi}{\beta}}-a^{\frac{2m\pi}{\beta}}}\left[\left(\frac{r}{a}\right)^{\frac{m\pi}{\beta}}-\left(\frac{a}{r}\right)^{\frac{m\pi}{\beta}}\right]a_m\sin\frac{m\pi\phi}{\beta}\right\}$$

where
$$a_m=\frac{2}{\beta}\int_0^\beta F(\phi)\sin\frac{m\pi\phi}{\beta}\,d\phi.$$

14. If $V=\chi(r)$ when $\phi=0$, $V=0$ when $\phi=\beta$, $V=0$ when $r=a$, and $V=0$ when $r=b$, then if $a<r<b$ and $0<\phi<\beta$

$$V=\sum_{m=1}^{m=\infty}\left\{a_m\frac{\sinh\dfrac{m\pi(\beta-\phi)}{\log b-\log a}}{\sinh\dfrac{m\pi\beta}{\log b-\log a}}\sin\frac{m\pi(\log r-\log a)}{\log b-\log a}\right\}$$

where $$a_m = \frac{2}{\log b - \log a}\int_0^{\log\frac{b}{a}} \chi(ae^x)\sin\frac{m\pi x}{\log b - \log a}\,dx.$$

15. If $V=\psi(r)$ when $\phi=\beta$, $V=0$ when $\phi=0$, $V=0$ when $r=$ and $V=0$ when $r=b$, then if $a<r<b$ and $0<\phi<\beta$

$$V=\sum_{m=1}^{m=\infty}\left\{a_m\frac{\sinh\dfrac{m\pi\phi}{\log b-\log a}}{\sinh\dfrac{m\pi\beta}{\log b-\log a}}\sin\frac{m\pi(\log r-\log a)}{\log b-\log a}\right\}$$

where $$a_m = \frac{2}{\log b - \log a}\int_0^{\log\frac{b}{a}} \psi(ae^x)\sin\frac{m\pi x}{\log b - \log a}\,dx.$$

II. *Potential Function in Space.*

1. Show that

$$f(x,y)=\frac{1}{\pi^2}\int_0^\infty d\alpha\int_0^\infty d\beta\int_{-\infty}^{\infty} d\lambda\int_{-\infty}^{\infty} f(\lambda,\mu)\cos\alpha(\lambda-x)\cos\beta(\mu-y).d\mu,$$

for all values of x and y.

2. Find particular solutions of $D_x^2V+D_y^2V+D_z^2V=0$ in the forms

$$V=e^{\pm z\sqrt{\alpha^2+\beta^2}}\cos(\alpha x+\beta y)$$

$$V=e^{\pm z\sqrt{\alpha^2+\beta^2}}\sin(\alpha x+\beta y)$$

$$V=\sinh z\sqrt{\alpha^2+\beta^2}.\sin(\alpha x+\beta y)$$

$$V=\cosh z\sqrt{\alpha^2+\beta^2}.\sin(\alpha x+\beta y)$$

&c.

3. Given $D_x^2V+D_y^2V+D_z^2V=0$, and $V=f(x,y)$ when $z=0$, solve positive values of z.

Result: $$V=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\int_{-\infty}^{\infty}\frac{zf(\lambda,\mu)d\mu}{[z^2+(\lambda-x)^2+(\mu-y)^2]^{\frac{3}{2}}}.$$

4. Confirm the result of the last example by showing that if $f(x,y)$ is in-pendent of y

$$V=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{zf(\lambda,\mu)d\lambda}{z^2+(\lambda-x)^2}$$ (v. Ex. 3 Art. 4

5. If $D_x^2V + D_y^2V + D_z^2V = 0$, and $V = 1$ when $z = 0$ for all points within the rectangle bounded by the lines $x = a$, $x = -a$, $y = b$, and $y = -b$; and $V = 0$ when $z = 0$ for all points outside of this rectangle, then

$$2\pi V = \frac{b-y}{\sqrt{(b-y)^2}}\left\{\frac{\pi}{2} + \frac{1}{2}\sin^{-1}\frac{(a-x)^2(b-y)^2 - z^2[(a-x)^2+(b-y)^2+z^2]}{(a-x)^2(b-y)^2 + z^2[(a-x)^2+(b-y)^2+z^2]}\right.$$
$$\left. + \frac{1}{2}\sin^{-1}\frac{(a+x)^2(b-y)^2 - z^2[(a+x)^2+(b-y)^2+z^2]}{(a+x)^2(b-y)^2 + z^2[(a+x)^2+(b-y)^2+z^2]}\right\}$$
$$+ \frac{b+y}{\sqrt{(b+y)^2}}\left\{\frac{\pi}{2} + \frac{1}{2}\sin^{-1}\frac{(a-x)^2(b+y)^2 - z^2[(a-x)^2+(b+y)^2+z^2]}{(a-x)^2(b+y)^2 + z^2[(a-x)^2+(b+y)^2+z^2]}\right.$$
$$\left. + \frac{1}{2}\sin^{-1}\frac{(a+x)^2(b+y)^2 - z^2[(a+x)^2+(b+y)^2+z^2]}{(a+x)^2(b+y)^2 + z^2[(a+x)^2+(b+y)^2+z^2]}\right\}$$

if $-a < x < a$, and

$$4\pi V = \frac{b-y}{\sqrt{(b-y)^2}}\left\{\sin^{-1}\frac{(a-x)^2(b-y)^2 - z^2[(a-x)^2+(b-y)^2+z^2]}{(a-x)^2(b-y)^2 + z^2[(a-x)^2+(b-y)^2+z^2]}\right.$$
$$\left. - \sin^{-1}\frac{(a+x)^2(b-y)^2 - z^2[(a+x)^2+(b-y)^2+z^2]}{(a+x)^2(b-y)^2 + z^2[(a+x)^2+(b-y)^2+z^2]}\right\}$$
$$+ \frac{b+y}{\sqrt{(b+y)^2}}\left\{\sin^{-1}\frac{(a-x)^2(b+y)^2 - z^2[(a-x)^2+(b+y)^2+z^2]}{(a-x)^2(b+y)^2 + z^2[(a-x)^2+(b+y)^2+z^2]}\right.$$
$$\left. - \sin^{-1}\frac{(a+x)^2(b+y)^2 - z^2[(a+x)^2+(b+y)^2+z^2]}{(a+x)^2(b+y)^2 + z^2[(a+x)^2+(b+y)^2+z^2]}\right\};$$

if $x < -a$ or $x > a$.

6. If the value of the potential function V is given at every point of the base of an infinite rectangular prism and if the sides of the prism are at potential zero the value of V at any point within the prism is

$$V = \frac{4}{ab}\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} e^{-\pi z\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\int_0^a d\lambda\int_0^b f(\lambda,\mu)\sin\frac{m\pi\lambda}{a}\sin\frac{n\pi\mu}{b}\,d\mu.$$

If $V = 1$ on the base of the prism this reduces to

$$V = \frac{16}{\pi^2}\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty} e^{-\pi z\sqrt{\frac{(2m+1)^2}{a^2}+\frac{(2n+1)^2}{b^2}}}\frac{\sin\frac{(2m+1)\pi x}{a}\sin\frac{(2n+1)\pi y}{b}}{(2m+1)(2n+1)}.$$

7. If the value of the potential function on five faces of a rectangular parallelopiped, whose length, breadth, and height are a, b, and c, is zero, and

if the value of V is given for every point of the sixth face, then for any point within the parallelopiped

$$V=\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}A_{m,n}\frac{\sinh \pi(c-z)\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}{\sinh \pi c\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

where $$A_{m,n}=\frac{4}{ab}\int_0^a d\lambda\int_0^b f(\lambda,\mu)\sin\frac{m\pi\lambda}{a}\sin\frac{n\pi\mu}{b}\,d\mu.$$

8. If the value of the potential function is given on two opposite faces of a rectangular parallelopiped and is zero on the four remaining faces, then within the parallelopiped

$$V=\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}A_{m,n}\frac{\sinh \pi(c-z)\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}{\sinh \pi c\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$+\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}B_{m,n}\frac{\sinh \pi z\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}{\sinh \pi c\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

where $$A_{m,n}=\frac{4}{ab}\int_0^a d\lambda\int_0^b f(\lambda,\mu)\sin\frac{m\pi\lambda}{a}\sin\frac{n\pi\mu}{b}\,d\mu$$

and $$B_{m,n}=\frac{4}{ab}\int_0^a d\lambda\int_0^b F(\lambda,\mu)\sin\frac{m\pi\lambda}{a}\sin\frac{n\pi\mu}{b}\,d\mu.$$

9. If the value of the potential function is given at every point on the surface of a rectangular parallelopiped, what is its value at any point within the parallelopiped?

III. *Conduction of Heat in a Plane.*

1. Find particular solutions of $D_t u = a^2(D_x^2 u + D_y^2 u)$ of the forms

$$u = e^{-a^2(\alpha^2+\beta^2)t}\sin(\alpha x+\beta y)$$

$$u = e^{-a^2(\alpha^2+\beta^2)t}\cos(\alpha x+\beta y).$$

2. Given the initial temperature of every point in a thin plane plate, find the temperature of any point at any time,

$$u = \frac{1}{4a^2\pi t}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} e^{-\frac{(\lambda - x)^2 + (\mu - y)^2}{4a^2 t}} f(\lambda, \mu) d\mu$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\beta^2} d\beta \int_{-\infty}^{\infty} e^{-\gamma^2} f(x + 2a\sqrt{t}.\beta,\ y + 2a\sqrt{t}.\gamma) d\gamma.$$

3. For an instantaneous *source* of strength Q at (λ, μ)

$$u = \frac{Q}{4\pi a^2 t} e^{-\frac{(\lambda - x)^2 + (\mu - y)^2}{4a^2 t}} \qquad \text{v. Art. 53.}$$

For an instantaneous *doublet* of strength P at $(0, \mu)$ with its axis perpendicular to the axis of Y

$$u = \frac{Px}{8\pi a^4 t^2} e^{-\frac{x^2 + (\mu - y)^2}{4a^2 t}} \qquad \text{v. Art. 54.}$$

For a permanent doublet of strength P at $(0, \mu)$ with its axis perpendicular to the axis of Y

$$u = \frac{P}{2\pi a^2} \frac{x}{x^2 + (\mu - y)^2} e^{-\frac{x^2 + (\mu - y)^2}{4a^2 t}}.$$

If the strength of the doublet were $Pd\mu$ and the heat were uniformly generated and absorbed along the element $d\mu$ of the axis of Y beginning at $(0, \mu)$ we should have

$$u = \frac{P}{2\pi a^2} e^{-\frac{x^2 + (\mu - y)^2}{4a^2 t}} \frac{x d\mu}{x^2 + (\mu - y)^2} = \frac{P}{2\pi a^2} e^{-\frac{x^2 + (\mu - y)^2}{4a^2 t}} d \tan^{-1}\frac{\mu - y}{x},$$

and since $d\tan^{-1}\frac{\mu - y}{x}$ is the angle ARA', where A and A' are the points $(0, \mu)$ and $(0, \mu + d\mu)$ and R is the point (x, y), $u = 0$ when $x = 0$ unless $\mu < y < \mu + d\mu$, in which case $u = \frac{P}{2a^2}$ if x approaches zero from the positive side; and $u = 0$ when $t = 0$ except in the element $d\mu$. If then $u = 0$ when $t = 0$ and $u = f(y)$ when $x = 0$ we have only to suppose a doublet of strength $2a^2 f(x) dx$ placed in each element of the axis of Y and then to integrate; we get

$$u = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\frac{x^2 + (\mu - y)^2}{4a^2 t}} \frac{x f(\mu)}{x^2 + (\mu - y)^2} d\mu.$$

For a permanent doublet of strength $F(t)$ at $(0, \mu)$ we have

$$u = \frac{x}{8\pi a^4}\int_0^t e^{-\frac{x^2 + (\mu - y)^2}{4a^2(t - \tau)}} (t - \tau)^{-2} F(\tau) d\tau$$

$$= \frac{1}{2\pi a^2}\left[\frac{xF(0)}{x^2 + (\mu - y)^2} e^{-\frac{x^2 + (\mu - y)^2}{4a^2 t}} + \int_0^t \frac{xF'(\tau)}{x^2 + (\mu - y)^2} e^{-\frac{x^2 + (\mu - y)^2}{4a^2(t - \tau)}} d\tau\right].$$

From the reasoning above this must be zero when $t=0$ except at the point $(0,\mu)$, must be $2a^2F(t)$ at the point $(0,\mu)$, and 0 at every other point of the axis of Y when t is not zero.

Hence if $u=0$ when $t=0$ and $u=F(y,t)$ when $x=0$

$$u=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{xF(\mu,0)}{x^2+(\mu-y)^2}e^{-\frac{x^2+(\mu-y)^2}{4a^2t}}d\mu+\frac{1}{\pi}\int_{-\infty}^{\infty}d\mu\int_0^t\frac{xD_\tau F(\mu,\tau)}{x^2+(\mu-y)^2}e^{-\frac{x^2+(\mu-y)^2}{4a^2(t-\tau)}}d\tau.$$

For an extension of this solution by the method of images to the case where there are other rectilinear boundaries and for its application to the corresponding problems in the flow of heat in three dimensions see E. W. Hobson in Vol. XIX Proc. Lond. Math. Soc.

4. If the perimeter of a thin plane rectangular plate is kept at the temperature zero and the initial temperatures of all points of the plate are given, then for any point of the plate

$$u=\frac{4}{bc}\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}e^{-a^2\pi^2\left(\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)t}\sin\frac{m\pi x}{b}\sin\frac{n\pi y}{c}\int_0^b d\lambda\int_0^c f(\lambda,\mu)\sin\frac{m\pi\lambda}{b}\sin\frac{n\pi\mu}{c}d\mu.$$

if b is the length and c the breadth of the plate.

5. A large mass of iron at the temperature $0°$ contains an iron core in the shape of a long prism 40 cm. square. The core is removed and heated to the temperature of $100°$ throughout and then replaced. Find the temperature of a point in the axis of the core fifteen minutes afterward. Given $a^2=.185$ in C.G.S. units.

Ans., $52°.9$.

6. If the prism described in Ex. 5 after being heated to $100°$ has its lateral faces kept for 15 minutes at the temperature $0°$ find the temperature of a point in its axis.

Ans., $20°.8$.

IV. *Conduction of Heat in Space.*

1. Show that

$$\frac{1}{\pi^3}\int_0^\infty d\alpha\int_0^\infty d\beta\int_0^\infty d\gamma\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}d\mu\int_{-\infty}^{\infty}f(\lambda,\mu,\nu)\cos\alpha(\lambda-x)\cos\beta(\mu-y)\cos\gamma(\nu-z)d\nu$$
$$=f(x,y,z)$$

for all values of x, y, and z.

2. Show that

$$f(x,y,z)=\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\sum_{p=1}^{p=\infty}A_{m,n,p}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{c}$$

where $$A_{m,n,p}=\frac{8}{abc}\int_0^a d\lambda\int_0^b d\mu\int_0^c f(\lambda,\mu,\nu)\sin\frac{m\pi\lambda}{a}\sin\frac{n\pi\mu}{b}\sin\frac{p\pi\nu}{c}d\nu,$$

for $0<x<a$, $0<y<b$, $0<z<c$.

3. Obtain particular solutions of $D_t u = a^2(D_x^2 u + D_y^2 u + D_z^2 u)$ of the forms

$$u = e^{-a^2(\alpha^2+\beta^2+\gamma^2)t}\sin(\alpha x \pm \beta y \pm \gamma z)$$

$$u = e^{-a^2(\alpha^2+\beta^2+\gamma^2)t}\cos(\alpha x \pm \beta y \pm \gamma z).$$

4 Given the initial temperature of every point in an infinite homogeneous solid find the temperature of any point at any time.

$$u = \frac{1}{8a^3(\pi t)^{\frac{3}{2}}}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu \int_{-\infty}^{\infty} e^{-\frac{(\lambda-x)^2+(\mu-y)^2+(\nu-z)^2}{4a^2t}} f(\lambda, \mu, \nu)d\nu$$

$$= \frac{1}{\pi^{\frac{3}{2}}}\int_{-\infty}^{\infty} e^{-\beta^2}d\beta \int_{-\infty}^{\infty} e^{-\gamma^2}d\gamma \int_{-\infty}^{\infty} e^{-\delta^2} f(x + 2a\sqrt{t}.\beta, y + 2a\sqrt{t}.\gamma, z + 2a\sqrt{t}.\delta)d\delta.$$

5. If the surface of a rectangular parallelopiped is kept at the temperature zero and the initial temperatures of all points of the parallelopiped are given, then for any point of the parallelopiped

$$u = \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\sum_{p=1}^{p=\infty} A_{m,n,p}\, e^{-a^2\pi^2\left(\frac{m^2}{b^2}+\frac{n^2}{c^2}+\frac{p^2}{d^2}\right)t}\sin\frac{m\pi x}{b}\sin\frac{n\pi y}{c}\sin\frac{p\pi z}{d}$$

where $$A_{m,n,p} = \frac{8}{bcd}\int_0^b d\lambda\int_0^c d\mu\int_0^d f(\lambda, \mu, \nu)\sin\frac{m\pi\lambda}{b}\sin\frac{n\pi\mu}{c}\sin\frac{p\pi\nu}{d}\,d\nu.$$

6. An iron cube 40 cm. on an edge is heated to the uniform temperature of 100° Centigrade and then tightly enclosed in a large iron mass which is at the uniform temperature of 0°. Find the temperature of the centre of the cube fifteen minutes afterwards. *Ans.*, 38°.4.

7. An iron cube 40 cm. on an edge is heated to the uniform temperature of 100° and then its surface is kept for fifteen minutes at the temperature 0°. Required the temperature of its centre. *Ans.*, 9°.5.

CHAPTER V.*

ZONAL HARMONICS.

74. In Art. 16 we obtained

$$z = Ap_m(x) + Bq_m(x) \tag{1}$$

[v. (6) Art. 16] as the general solution of Legendre's Equation

$$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + m(m+1)z = 0, \tag{2}$$

m being wholly unrestricted in value and x lying between -1 and 1; where

$$p_m(x) = 1 - \frac{m(m+1)}{2!}x^2 + \frac{m(m-2)(m+1)(m+3)}{4!}x^4$$
$$- \frac{m(m-2)(m-4)(m+1)(m+3)(m+5)}{6!}x^6 + \cdots \tag{3}$$

and

$$q_m(x) = x - \frac{(m-1)(m+2)}{3!}x^3 + \frac{(m-1)(m-3)(m+2)(m+4)}{5!}x^5$$
$$- \frac{(m-1)(m-3)(m-5)(m+2)(m+4)(m+6)}{7!}x^7 + \cdots; \tag{4}$$

and we found

$$\left.\begin{aligned} V &= r^m p_m(\cos\theta) \\ V &= \frac{1}{r^{m+1}}p_m(\cos\theta) \\ V &= r^m q_m(\cos\theta) \\ V &= \frac{1}{r^{m+1}}q_m(\cos\theta), \end{aligned}\right\} \tag{5}$$

m being unrestricted in value, as particular solutions of the special form assumed by Laplace's Equation in spherical coördinates when V is independent of ϕ; that is, of the equation

$$rD_r^2(rV) + \frac{1}{\sin\theta}D_\theta(\sin\theta\, D_\theta V) = 0. \tag{6}$$

* Before reading this chapter the student is advised to re-read carefully articles 9, 10, 13(c), 15, 16, and 18(c).

For the important case where m is a positive integer we found

$$z = A\,P_m(x) + B\,Q_m(x) \tag{7}$$

[v. (10) Art. 16] as the general solution of Legendre's Equation (2), whence

$$\left.\begin{aligned} V &= r^m P_m(\cos\theta) \\ V &= \frac{1}{r^{m+1}} P_m(\cos\theta) \\ V &= r^m Q_m(\cos\theta) \\ V &= \frac{1}{r^{m+1}} Q_m(\cos\theta) \end{aligned}\right\} \tag{8}$$

are particular solutions of (6) if m is a positive integer.

$$P_m(x) = \frac{(2m-1)(2m-3)\cdots 1}{m!}\left[x^m - \frac{m(m-1)}{2(2m-1)}x^{m-2} + \frac{m(m-1)(m-2)(m-3)}{2.4.(2m-1)(2m-3)}x^{m-4} - \cdots\right] \tag{9}$$

[v. (8) Art. 16] and is a finite sum terminating with the term which involves x if m is odd and with the term involving x^0 if m is even.

It is called a *Surface Zonal Harmonic*, or a *Legendre's Coefficient*, or more briefly a *Legendrian*.

$$Q_m(x) = \frac{m!}{(2m+1)(2m-1)\cdots 1}\left[\frac{1}{x^{m+1}} + \frac{(m+1)(m+2)}{2.(2m+3)}\frac{1}{x^{m+3}} + \frac{(m+1)(m+2)(m+3)(m+4)}{2.4.(2m+3)(2m+5)}\frac{1}{x^{m+5}} + \cdots\right] \tag{10}$$

if $x < -1$ or $x > 1$. [v. (9) Art. 16.]

It is called a *Surface Zonal Harmonic* of the *second kind*.

$$\begin{aligned} Q_m(x) &= (-1)^{\frac{m+1}{2}} \frac{2^{m-1}\left[\Gamma\left(\frac{m+1}{2}\right)\right]^2}{\Gamma(m+1)} p_m(x) \\ &= (-1)^{\frac{m+1}{2}} \frac{2.4.6.\cdots(m-1)}{3.5.7.\cdots m} p_m(x) \end{aligned} \tag{11}$$

[v. (13) Art. 16] if m is odd and $-1 < x < 1$.

$$\begin{aligned} Q_m(x) &= (-1)^{\frac{m}{2}} \frac{2^m\left[\Gamma\left(\frac{m}{2}+1\right)\right]^2}{\Gamma(m+1)} q_m(x) \\ &= (-1)^{\frac{m}{2}} \frac{2.4.6.\cdots m}{1.3.5.\cdots(m-1)} q_m(x) \end{aligned} \tag{12}$$

[v. (14) Art. 16] if m is even and $-1 < x < 1$.

In most of the work that immediately follows we shall regard x in $P_m(x)$ as equal to $\cos\theta$ and therefore as lying between -1 and 1.*

75. In Article 9 the undetermined coefficient a_m of x^m in $P_m(x)$ was arbitrarily written in the form $\dfrac{(2m-1)(2m-3)\cdots 1}{m!}$ for reasons which shall now be given.

In Articles 9 and 16 $z = P_m(x)$ was obtained as a particular solution of Legendre's Equation

$$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + m(m+1)z = 0 \tag{1}$$

by the device of assuming that z could be expressed as a sum or a series of terms of the form $a_n x^n$ and then determining the coefficients. We can, however, obtain a particular solution of Legendre's Equation by an entirely different method.

The potential function due to a unit of mass concentrated at a given point (x_1, y_1, z_1) is

$$V = \frac{1}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}} \tag{2}$$

and this must be a particular solution of Laplace's Equation

$$D_x^2 V + D_y^2 V + D_z^2 V = 0, \tag{3}$$

as is easily verified by direct substitution.

If we transform (2) to spherical coordinates using the formulas of transformation

$$x = r\cos\theta$$

$$y = r\sin\theta\cos\phi$$

$$z = r\sin\theta\sin\phi$$

we get

$$V = \frac{1}{\sqrt{r^2 - 2rr_1[\cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi-\phi_1)] + r_1^2}} \tag{4}$$

as a solution of Laplace's Equation in Spherical Coordinates

$$rD_r^2(rV) + \frac{1}{\sin\theta}D_\theta(\sin\theta\, D_\theta V) + \frac{1}{\sin^2\theta}D_\phi^2 V = 0 \quad \text{[XIII] Art. 1.}$$

If the given point (x_1, y_1, z_1) is taken on the axis of X, as it must be that (4) may be independent of ϕ, $\theta_1 = 0$, and

$$V = \frac{1}{\sqrt{r^2 - 2rr_1\cos\theta + r_1^2}} \tag{5}$$

* English writers on Spherical Harmonics generally use μ in place of x for $\cos\theta$. We shall follow them, however, only when we should thereby avoid confusion.

is a solution of

$$rD_r^2(rV)+\frac{1}{\sin\theta}D_\theta(\sin\theta D_\theta V)=0. \tag{6}$$

Equation (5) may be written

$$V=\frac{1}{r}\frac{1}{\sqrt{1-2\frac{r_1}{r}\cos\theta+\frac{r_1^2}{r^2}}} \tag{7}$$

or

$$V=\frac{1}{r_1}\frac{1}{\sqrt{1-2\frac{r}{r_1}\cos\theta+\frac{r^2}{r_1^2}}}. \tag{8}$$

$\sqrt{1-2z\cos\theta+z^2}$ is finite and continuous for all values real or complex of z. It is double-valued but the two branches of the function are distinct except for the values of z which make $1-2z\cos\theta+z^2=0$ namely $z=\cos\theta+i\sin\theta$ and $z=\cos\theta-i\sin\theta$, both of which have the modulus unity and which are *critical* values.

$\frac{1}{\sqrt{1-2z\cos\theta+z^2}}$ is finite and continuous except for the values of $z=\cos\theta-i\sin\theta$ and $z=\cos\theta+i\sin\theta$ for which it becomes infinite; it is double-valued but has as critical values only these values of z. It is then *holomorphic* within a circle described with the origin as centre and the radius unity, and can be developed into a power series which will be convergent for all values of z having moduli less than one. (Int. Cal. Arts. 207, 212, 214, 220.)

If then $r>r_1$ $\frac{1}{\sqrt{1-\frac{2r_1}{r}\cos\theta+\frac{r_1^2}{r^2}}}$ can be developed into a convergent series involving whole powers of $\frac{r_1}{r}$.

Let $\sum p_m\frac{r_1^m}{r^m}$ be this series, p_m, of course, being a function of $\cos\theta$. Then

$$V=\frac{1}{r}\sum p_m\frac{r_1^m}{r^m}$$

[v. (7)] is a solution of (6). Substitute this value of V in (6) and we get

$$\sum\left[\frac{r_1^m}{r^{m+1}}m(m+1)p_m+\frac{r_1^m}{r^{m+1}}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dp_m}{d\theta}\right)\right]=0.$$

As this must hold whatever the value of r provided $r>r_1$ the coefficient of each power of r must be zero, and hence the equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dp_m}{d\theta}\right)+m(m+1)p_m=0 \tag{9}$$

must be true.

But as we have seen in Art. 9 the substitution of $x = \cos\theta$ in (9) re-
it to

$$(1-x^2)\frac{d^2p_m}{dx^2} - 2x\frac{dp_m}{dx} + m(m+1)p_m = 0,$$

and therefore $z = p_m$

is a solution of Legendre's Equation (1).

If $r < r_1$, $\dfrac{1}{\sqrt{1 - \dfrac{2r}{r_1}\cos\theta + \dfrac{r^2}{r_1^2}}}$ can be developed into a convergent

involving whole powers of $\dfrac{r}{r_1}$.

Let $\sum p_m \dfrac{r^m}{r_1^m}$ be this series. Then

$$V = \frac{1}{r_1}\sum p_m \frac{r^m}{r_1^m}$$

(v. 8) is a solution of (6); substituting in (6) we get

$$\sum\left[\frac{r^m}{r_1^{m+1}}m(m+1)p_m + \frac{r^m}{r_1^{m+1}}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dp_m}{d\theta}\right)\right] = 0,$$

whence it follows as before that

$$z = p_m$$

is a solution of Legendre's Equation.

But p_m is the coefficient of the mth power of $\dfrac{r}{r_1}$ in the developm
$\left(1 - 2\dfrac{r}{r_1}\cos\theta + \dfrac{r^2}{r_1^2}\right)^{-\frac{1}{2}}$ according to powers of $\dfrac{r}{r_1}$, or of the mth power o
the development of $\left(1 - 2\dfrac{r_1}{r}\cos\theta + \dfrac{r_1^2}{r^2}\right)^{-\frac{1}{2}}$ according to powers of

more briefly it is the coefficient of the mth power of z in the developm
$(1 - 2xz + z^2)^{-\frac{1}{2}}$ according to powers of z, x standing for $\cos\theta$.

$$(1 - 2xz + z^2)^{-\frac{1}{2}} = [1 - z(2x - z)]^{-\frac{1}{2}}$$

and can be developed by the Binomial Theorem; the coefficient of z^m is
picked out and is

$$\frac{(2m-1)(2m-3)\cdots 1}{m!}\left[x^m - \frac{m(m-1)}{2(2m-1)}x^{m-2}\right.$$

$$\left.+ \frac{m(m-1)(m-2)(m-3)}{2\cdot 4(2m-1)(2m-3)}x^{m-4} - \cdots\right.$$

But this is precisely $P_m(x)$. [v. Art. 74 (3).]

Hence $P_m(x)$ is equal to the coefficient of the mth power o
the development of $[1 - 2xz + z^2]^{-\frac{1}{2}}$ into a power series, the modulu
being less than unity.

76. If $x=1$ $P_m(x)=1$. For if $x=1$ $(1-2xz+z^2)^{-\frac{1}{2}}$ reduces to $(1-2z+z^2)^{-\frac{1}{2}}$ that is to $(1-z)^{-1}$, which develops into

$$1+z+z^2+z^3+z^4+\cdots,$$

and the coefficient of each power of z is unity. Therefore

$$P_m(1)=1. \tag{1}$$

We have seen that if m is even $P_m(x)$ contains only even powers of x and terminates with the term involving x^0, that is with the constant term.

The value of this constant term can be picked out from the formula for $P_m(x)$ [v. Art. 74 (9)]. It is $(-1)^{\frac{m}{2}}\dfrac{1.3.5.\cdots(m-1)}{2.4.6.\cdots m}$; or it can be found as follows:— It is clearly the value $P_m(x)$ assumes when $x=0$; it is, then, the coefficient of z^m in the development of $(1+z^2)^{-\frac{1}{2}}$; but

$$(1+z^2)^{-\frac{1}{2}}=1-\frac{1}{2}z^2+\frac{1.3}{2.4}z^4-\frac{1.3.5}{2.4.6}z^6+\frac{1.3.5.7}{2.4.6.8}z^8-\cdots$$

and the coefficient of z^m, m being an even number, is $(-1)^{\frac{m}{2}}\dfrac{1.3.5\cdots(m-1)}{2.4.6\cdots m}$.

If m is odd $P_m(x)$ contains only odd powers of x and terminates with the term involving x to the first power. The coefficient of this term can be picked out from (9) Art. 74 and is $(-1)^{\frac{m-1}{2}}\dfrac{3.5.7.\cdots m}{2.4.6.\cdots(m-1)}$; or it can be found as follows: It is clearly the value assumed by $\dfrac{dP_m(x)}{dx}$ when $x=0$.

It is, then, the coefficient of z^m in the development of $\dfrac{z}{(1+z^2)^{\frac{3}{2}}}$.

$$\frac{z}{(1+z^2)^{\frac{3}{2}}}=z-\frac{3}{2}z^3+\frac{3.5}{2.4}z^5-\frac{3.5.7}{2.4.6}z^7+\cdots$$

and the coefficient of z^m in this development is $(-1)^{\frac{m-1}{2}}\dfrac{3.5.7\cdots m}{2.4.6\cdots(m-1)}$, m being an odd number.

77. To recapitulate:

$$P_m(x)=\frac{1.3.5\cdots(2m-1)}{m!}\Big[x^m-\frac{m(m-1)}{2(2m-1)}x^{m-2}$$
$$+\frac{m(m-1)(m-2)(m-3)}{2.4.(2m-1)(2m-3)}x^{m-4}$$
$$-\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{2.4.6.(2m-1)(2m-3)(2m-5)}x^{m-6}+\cdots\Big], \tag{1}$$

But as we have seen in Art. 9 the substitution of $x = \cos\theta$ in (9) reduces it to

$$(1-x^2)\frac{d^2 p_m}{dx^2} - 2x\frac{dp_m}{dx} + m(m+1)\,p_m = 0,$$

and therefore
$$z = p_m$$
is a solution of Legendre's Equation (1).

If $r < r_1$ $\dfrac{1}{\sqrt{1 - \dfrac{2r}{r_1}\cos\theta + \dfrac{r^2}{r_1^2}}}$ can be developed into a convergent series involving whole powers of $\dfrac{r}{r_1}$.

Let $\sum p_m \dfrac{r^m}{r_1^m}$ be this series. Then

$$V = \frac{1}{r_1}\sum p_m \frac{r^m}{r_1^m}$$

(v. 8) is a solution of (6); substituting in (6) we get

$$\sum\left[\frac{r^m}{{r_1}^{m+1}}m(m+1)p_m + \frac{r^m}{{r_1}^{m+1}}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dp_m}{d\theta}\right)\right] = 0,$$

whence it follows as before that
$$z = p_m$$
is a solution of Legendre's Equation.

But p_m is the coefficient of the mth power of $\dfrac{r}{r_1}$ in the development of $\left(1 - 2\dfrac{r}{r_1}\cos\theta + \dfrac{r^2}{r_1^2}\right)^{-\frac{1}{2}}$ according to powers of $\dfrac{r}{r_1}$, or of the mth power of $\dfrac{r_1}{r}$ in the development of $\left(1 - 2\dfrac{r_1}{r}\cos\theta + \dfrac{r_1^2}{r^2}\right)^{-\frac{1}{2}}$ according to powers of $\dfrac{r_1}{r}$, or more briefly it is the coefficient of the mth power of z in the development of $(1 - 2xz + z^2)^{-\frac{1}{2}}$ according to powers of z, x standing for $\cos\theta$.

$$(1 - 2xz + z^2)^{-\frac{1}{2}} = [1 - z(2x - z)]^{-\frac{1}{2}}$$

and can be developed by the Binomial Theorem; the coefficient of z^m is easily picked out and is

$$\frac{(2m-1)(2m-3)\cdots 1}{m!}\left[x^m - \frac{m(m-1)}{2(2m-1)}x^{m-2} + \frac{m(m-1)(m-2)(m-3)}{2.4.(2m-1)(2m-3)}x^{m-4} - \cdots\right].$$

But this is precisely $P_m(x)$. [v. Art. 74 (9)]

Hence $P_m(x)$ is equal to the coefficient of the mth power of z in the development of $[1 - 2xz + z^2]^{-\frac{1}{2}}$ into a power series, the modulus of z being less than unity.

76. If $x = 1$ $P_m(x) = 1$. For if $x = 1$ $(1 - 2xz + z^2)^{-\frac{1}{2}}$ reduces to $(1 - 2z + z^2)^{-\frac{1}{2}}$ that is to $(1 - z)^{-1}$, which develops into

$$1 + z + z^2 + z^3 + z^4 + \cdots,$$

and the coefficient of each power of z is unity. Therefore

$$P_m(1) = 1. \tag{1}$$

We have seen that if m is even $P_m(x)$ contains only even powers of x and terminates with the term involving x^0, that is with the constant term.

The value of this constant term can be picked out from the formula for $P_m(x)$ [v. Art. 74 (9)]. It is $(-1)^{\frac{m}{2}} \dfrac{1.3.5.\cdots(m-1)}{2.4.6.\cdots m}$; or it can be found as follows:—It is clearly the value $P_m(x)$ assumes when $x = 0$; it is, then, the coefficient of z^m in the development of $(1 + z^2)^{-\frac{1}{2}}$; but

$$(1 + z^2)^{-\frac{1}{2}} = 1 - \frac{1}{2} z^2 + \frac{1.3}{2.4} z^4 - \frac{1.3.5}{2.4.6} z^6 + \frac{1.3.5.7}{2.4.6.8} z^8 - \cdots$$

and the coefficient of z^m, m being an even number, is $(-1)^{\frac{m}{2}} \dfrac{1.3.5\cdots(m-1)}{2.4.6\cdots m}$.

If m is odd $P_m(x)$ contains only odd powers of x and terminates with the term involving x to the first power. The coefficient of this term can be picked out from (9) Art. 74 and is $(-1)^{\frac{m-1}{2}} \dfrac{3.5.7.\cdots m}{2.4.6.\cdots(m-1)}$; or it can be found as follows:—It is clearly the value assumed by $\dfrac{dP_m(x)}{dx}$ when $x = 0$. It is, then, the coefficient of z^m in the development of $\dfrac{z}{(1+z^2)^{\frac{3}{2}}}$.

$$\frac{z}{(1+z^2)^{\frac{3}{2}}} = z - \frac{3}{2} z^3 + \frac{3.5}{2.4} z^5 - \frac{3.5.7}{2.4.6} z^7 + \cdots$$

and the coefficient of z^m in this development is $(-1)^{\frac{m-1}{2}} \dfrac{3.5.7\cdots m}{2.4.6\cdots(m-1)}$, m being an odd number.

77. To recapitulate:

$$P_m(x) = \frac{1.3.5\cdots(2m-1)}{m!} \left[x^m - \frac{m(m-1)}{2(2m-1)} x^{m-2} \right.$$
$$+ \frac{m(m-1)(m-2)(m-3)}{2.4.(2m-1)(2m-3)} x^{m-4}$$
$$\left. - \frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{2.4.6.(2m-1)(2m-3)(2m-5)} x^{m-6} + \cdots \right], \tag{1}$$

m being a positive integer, is a *Surface Zonal Harmonic* or *Legendrian* of the mth order. It is a finite sum terminating with the first power of x if m is odd, and with the zeroth power of x if m is even.

$P_m(x)$ is the coefficient of the mth power of z in the development of $(1-2xz+z^2)^{-\frac{1}{2}}$ into a power series. Hence if $z<1$

$$(1-2xz+z^2)^{-\frac{1}{2}} = P_0(x) + P_1(x).z + P_2(x).z^2 + P_3(x).z^3 + P_4(x).z^4 + P_5(x).z^5 + \cdots + P_m(x).z^m + \cdots. \qquad (2)$$

Whence

$$\left.\begin{aligned} \frac{1}{\sqrt{r^2-2rr_1\cos\theta+r_1^2}} &= \frac{1}{r}\left[P_0(\cos\theta) + \frac{r_1}{r}P_1(\cos\theta) + \frac{r_1^2}{r^2}P_2(\cos\theta) + \cdots + \frac{r_1^m}{r^m}P_m(\cos\theta) + \cdots\right] \text{ if } r > r_1 \\ &= \frac{1}{r_1}\left[P_0(\cos\theta) + \frac{r}{r_1}P_1(\cos\theta) + \frac{r^2}{r_1^2}P_2(\cos\theta) + \cdots + \frac{r^m}{r_1^m}P_m(\cos\theta) + \cdots\right] \text{ if } r < r_1. \end{aligned}\right\} \quad (3)$$

$$z = P_m(x)$$

is a solution of Legendre's Equation

$$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + m(m+1)z = 0$$

when m is a positive integer.

$$V = r^m P_m(\cos\theta)$$

and

$$V = \frac{1}{r^{m+1}} P_m(\cos\theta)$$

are solutions of the form of Laplace's Equation in Spherical Coördinates which is independent of ϕ, namely

$$rD_r^2(rV) + \frac{1}{\sin\theta}D_\theta(\sin\theta\, D_\theta V) = 0. \qquad (4)$$

$$P_m(1) = 1. \qquad (5)$$

$$P_{2m}(-x) = P_{2m}(x). \qquad (6)$$

$$P_{2m+1}(-x) = -P_{2m+1}(x). \qquad (7)$$

$$P_{2m+1}(0) = 0. \qquad (8)$$

$$P_{2m}(0) = (-1)^m \frac{1.3.5.\cdots(2m-1)}{2.4.6.\cdots 2m}. \qquad (9)$$

$$\left[\frac{dP_{2m+1}(x)}{dx}\right]_{x=0} = (-1)^m \frac{3.5.7.\cdots(2m+1)}{2.4.6.\cdots 2m}. \tag{10}$$

For convenience of reference we write out a few Zonal Harmonics. They are obtained by substituting successive integers for m in formula (1).

$$\left.\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= \frac{1}{2}(3x^2-1)\\
P_3(x) &= \frac{1}{2}(5x^3-3x)\\
P_4(x) &= \frac{1}{8}(35x^4-30x^2+3)\\
P_5(x) &= \frac{1}{8}(63x^5-70x^3+15x)\\
P_6(x) &= \frac{1}{16}(231x^6-315x^4+105x^2-5)\\
P_7(x) &= \frac{1}{16}(429x^7-693x^5+315x^3-35x)\\
P_8(x) &= \frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35).
\end{aligned}\right\} \tag{11}$$

Any Surface Zonal Harmonic may be obtained from the two of next lower orders by the aid of the formula

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \tag{12}$$

which is easily obtained and is convenient when the numerical value of x is given.

Differentiate (2) with respect to z and we get

$$\frac{-(z-x)}{(1-2xz+z^2)^{\frac{3}{2}}} = P_1(x) + 2P_2(x).z + 3P_3(x).z^2 + \cdots$$

whence

$$\frac{-(z-x)}{(1-2xz+z^2)^{\frac{1}{2}}} = (1-2xz+z^2)(P_1(x)+2P_2(x).z+3P_3(x).z^2+\cdots).$$

Hence by (2)

$$(1-2xz+z^2)(P_1(x)+2P_2(x).z+3P_3(x).z^2+\cdots)$$
$$+(z-x)(P_0(x)+P_1(x).z+P_2(x).z^2+\cdots) = 0 \tag{13}$$

(13) is identically true, hence the coefficient of each power of z must
Picking out the coefficient of z^n and writing it equal to zero we have f
(12) above.*

78. We are now able to solve completely the problem considered in
We were to find a solution of the differential equation

$$rD_r^2(rV)+\frac{1}{\sin\theta}D_\theta(\sin\theta\, D_\theta V)=0$$

subject to the condition

$$V=\frac{M}{(c^2+r^2)^{\frac{1}{2}}}\quad\text{when}\quad\theta=0.$$

We know (v. Art. 77) that

$$V=r^m P_m(\cos\theta)$$

and

$$V=\frac{1}{r^{m+1}}P_m(\cos\theta)$$

are solutions of (1).

For values of $r<c$

$$\frac{M}{(c^2+r^2)^{\frac{1}{2}}}=\frac{M}{c}\left[1-\frac{1}{2}\frac{r^2}{c^2}+\frac{1.3}{2.4}\frac{r^4}{c^4}-\frac{1.3.5}{2.4.6}\frac{r^6}{c^6}+\cdots\right].$$

Therefore for values of $r<c$

$$V=\frac{M}{c}\left[P_0(\cos\theta)-\frac{1}{2}\frac{r^2}{c^2}P_2(\cos\theta)\right.$$
$$\left.+\frac{1.3}{2.4}\frac{r^4}{c^4}P_4(\cos\theta)-\frac{1.3.5}{2.4.6}\frac{r^6}{c^6}P_6(\cos\theta)+\cdots\right]$$

is our required solution; because each term satisfies equation (1), and
fore the whole value satisfies (1), and when $\theta=0$

$$P_m(\cos\theta)=P_m(1)=1$$

[v. (5) Art. 77], and hence (4) reduces to (3) and (2) is satisfied.

For values of $r>c$

$$\frac{M}{(c^2+r^2)^{\frac{1}{2}}}=\frac{M}{r}\left[1-\frac{1}{2}\frac{c^2}{r^2}+\frac{1.3}{2.4}\frac{c^4}{r^4}-\frac{1.3.5}{2.4.6}\frac{c^6}{r^6}+\cdots\right]$$

$$=M\left[\frac{1}{r}-\frac{1}{2}\frac{c^2}{r^3}+\frac{1.3}{2.4}\frac{c^4}{r^5}-\frac{1.3.5}{2.4.6}\frac{c^6}{r^7}+\cdots\right].$$

* For tables of Surface Zonal Harmonics v. Appendix Tables I and II.

Therefore for values of $r > c$

$$V = \frac{M}{c}\left[\frac{c}{r} P_0(\cos\theta) - \frac{1}{2}\frac{c^3}{r^3} P_2(\cos\theta) + \frac{1.3}{2.4}\frac{c^5}{r^5} P_4(\cos\theta) - \frac{1.3.5}{2.4.6}\frac{c^7}{r^7} P_6(\cos\theta) + \cdots\right] \tag{6}$$

is our required solution. For it satisfies (1) and reduces to (2) when $\theta = 0$.

79. As another example let us suppose a conductor in the form of a thin circular disc charged with electricity, and let it be required to find the value of the potential function at any point in space.

If the magnitude of the charge is M and the radius of the plate is a the surface density at a point of the plate at a distance r from the centre is

$$\sigma = \frac{M}{4a\pi\sqrt{a^2 - r^2}}$$

and all points of the conductor are at the potential $\frac{\pi M}{2a}$. (v. Peirce's Newtonian Potential Function, § 61.)

The value of the potential function at a point in the axis of the plate at the distance x from the plate is easily seen to be

$$V = \frac{M}{a}\int_0^a \frac{rdr}{\sqrt{(a^2 - r^2)(x^2 + r^2)}}$$

$$= \frac{M}{2a}\cos^{-1}\frac{x^2 - a^2}{x^2 + a^2}.$$

$$\frac{d}{dx}\left(\frac{M}{2a}\cos^{-1}\frac{x^2 - a^2}{x^2 + a^2}\right) = -\frac{M}{a^2 + x^2}$$

$$= -\frac{M}{a^2}\left[1 - \frac{x^2}{a^2} + \frac{x^4}{a^4} - \frac{x^6}{a^6} + \cdots\right]$$

if $x < a$,

$$= -\frac{M}{x^2}\left[1 - \frac{a^2}{x^2} + \frac{a^4}{x^4} - \frac{a^6}{x^6} + \cdots\right]$$

if $x > a$.

Integrating and then determining the arbitrary constant we have

$$\frac{M}{2a}\cos^{-1}\frac{x^2 - a^2}{x^2 + a^2} = \frac{M}{a}\left[\frac{\pi}{2} - \frac{x}{a} + \frac{x^3}{3a^3} - \frac{x^5}{5a^5} + \frac{x^7}{7a^7} - \cdots\right]$$

if $x < a$,

$$= \frac{M}{a}\left[\frac{a}{x} - \frac{a^3}{3x^3} + \frac{a^5}{5x^5} - \frac{a^7}{7x^7} + \cdots\right]$$

if $x > a$.

We have, then, to solve the equation

$$rD_r^2(rV)+\frac{1}{\sin\theta}D_\theta(\sin\theta\, D_\theta V)=0$$

subject to the conditions

$$V=\frac{M}{a}\left[\frac{\pi}{2}-\frac{r}{a}+\frac{r^3}{3a^3}-\frac{r^5}{5a^5}+\frac{r^7}{7a^7}-\cdots\right]$$

when $\theta=0$ and $r<a$

and
$$V=\frac{M}{a}\left[\frac{a}{r}-\frac{a^3}{3r^3}+\frac{a^5}{5r^5}-\frac{a^7}{7r^7}+\cdots\right]$$

when $\theta=0$ and $r>a$.

The required solution is easily seen to be

$$V=\frac{M}{a}\left[\frac{\pi}{2}-\frac{r}{a}P_1(\cos\theta)+\frac{1}{3}\frac{r^3}{a^3}P_3(\cos\theta)-\frac{1}{5}\frac{r^5}{a^5}P_5(\cos\theta)+\cdots\right]$$

if $r<a$ and $\theta<\frac{\pi}{2}$,

and $$V=\frac{M}{a}\left[\frac{a}{r}-\frac{1}{3}\frac{a^3}{r^3}P_2(\cos\theta)+\frac{1}{5}\frac{a^5}{r^5}P_4(\cos\theta)-\frac{1}{7}\frac{a^7}{r^7}P_6(\cos\theta)+\cdots\right]$$

if $r>a$.

EXAMPLES.

1. Given that if a charge M of electricity is placed on an ellipsoidal conductor the surface density at any point P of the conductor is equal to $\frac{Mp}{4\pi abc}$, where p is the distance from the centre of the conductor to the tangent plane at P (v. Peirce, New. Pot. Func. § 61); find the value of the potential function at any external point when the conductor is the oblate spheroid generated by the rotation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its minor axis.

Ans. (1) If the point is on the axis of revolution

$$V=\frac{M}{2\sqrt{a^2-b^2}}\left[\sin^{-1}\left(\frac{bx+a^2-b^2}{a\sqrt{x^2+a^2-b^2}}\right)-\sin^{-1}\left(\frac{bx-a^2+b^2}{a\sqrt{x^2+a^2-b^2}}\right)\right]$$

x being the distance from the centre.

(2) If the point is on the surface of the spheroid

$$V=\frac{M}{2\sqrt{a^2-b^2}}\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{2b^2-a^2}{a^2}\right)\right]=\frac{M}{\sqrt{a^2-b^2}}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{b}{\sqrt{a^2-b^2}}\right)\right].$$

(3) If the distance r of the point from the centre is less than $\sqrt{a^2-b^2}$ and $\theta<\frac{\pi}{2}$

$$V=\frac{M}{\sqrt{a^2-b^2}}\left[\frac{\pi}{2}-\frac{r}{(a^2-b^2)^{\frac{1}{2}}}P_1(\cos\theta)\right.$$
$$\left.+\frac{r^3}{3(a^2-b^2)^{\frac{3}{2}}}P_3(\cos\theta)-\frac{r^5}{5(a^2-b^2)^{\frac{5}{2}}}P_5(\cos\theta)+\cdots\right].$$

(4) If the distance r of the point from the centre is greater than $\sqrt{a^2-b^2}$

$$V=\frac{M}{\sqrt{a^2-b^2}}\left[\frac{(a^2-b^2)^{\frac{1}{2}}}{r}-\frac{(a^2-b^2)^{\frac{3}{2}}}{3r^3}P_2(\cos\theta)\right.$$
$$\left.+\frac{(a^2-b^2)^{\frac{5}{2}}}{5r^5}P_4(\cos\theta)-\frac{(a^2-b^2)^{\frac{7}{2}}}{7r^7}P_6(\cos\theta)+\cdots\right].$$

2. If the conductor is the prolate spheroid generated by the rotation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its major axis, show that if the point is an external point and is on the axis at a distance x from the centre,

$$V=\frac{M}{2\sqrt{a^2-b^2}}\log\frac{x+\sqrt{a^2-b^2}}{x-\sqrt{a^2-b^2}}.$$

If the point is not on the axis and $r>\sqrt{a^2-b^2}$

$$V=\frac{M}{\sqrt{a^2-b^2}}\left[\frac{(a^2-b^2)^{\frac{1}{2}}}{r}+\frac{(a^2-b^2)^{\frac{3}{2}}}{3r^3}P_2(\cos\theta)\right.$$
$$\left.+\frac{(a^2-b^2)^{\frac{5}{2}}}{5r^5}P_4(\cos\theta)+\frac{(a^2-b^2)^{\frac{7}{2}}}{7r^7}P_6(\cos\theta)+\cdots\right].$$

80. As a third example we will find the value of the potential function due to a thin homogeneous circular disc, of density ρ, thickness k, and radius a.

The value of V at a point in the axis of the disc at a distance x from its centre is readily found and proves to be

$$V_0=2\pi\rho k(\sqrt{x^2+a^2}-x)=\frac{2M}{a^2}[\sqrt{x^2+a^2}-x].$$

If $x>a$

$$\sqrt{x^2+a^2}=x\left(1+\frac{a^2}{x^2}\right)^{\frac{1}{2}}=x\left[1+\frac{1}{2}\frac{a^2}{x^2}-\frac{1.1}{2.4}\frac{a^4}{x^4}+\frac{1.1.3}{2.4.6}\frac{a^6}{x^6}-\frac{1.1.3.5}{2.4.6.8}\frac{a^8}{x^8}+\cdots\right.$$

and $$V_0=\frac{2M}{a}\left[\frac{1}{2}\frac{a}{x}-\frac{1.1}{2.4}\frac{a^3}{x^3}+\frac{1.1.3}{2.4.6}\frac{a^5}{x^5}-\frac{1.1.3.5}{2.4.6.8}\frac{a^7}{x^7}+\cdots\right].$$

If $x < a$

$$\sqrt{x^2+a^2} = a\left(1+\frac{x^2}{a^2}\right)^{\frac{1}{2}} = a\left[1+\frac{1}{2}\frac{x^2}{a^2}-\frac{1.1}{2.4}\frac{x^4}{a^4}+\frac{1.1.3}{2.4.6}\frac{x^6}{a^6}+\cdots\right]$$

and $$V_0 = \frac{2M}{a}\left[1-\frac{x}{a}+\frac{1}{2}\frac{x^2}{a^2}-\frac{1.1}{2.4}\frac{x^4}{a^4}+\frac{1.1.3}{2.4.6}\frac{x^6}{a^6}-\frac{1.1.3.5}{2.4.6.8}\frac{x^8}{a^8}+\cdots\right].$$

Hence the solution for any external point is

$$V = \frac{2M}{a}\left[\frac{1}{2}\frac{a}{r}-\frac{1.1}{2.4}\frac{a^3}{r^3}P_2(\cos\theta)\right.$$
$$\left.+\frac{1.1.3}{2.4.6}\frac{a^5}{r^5}P_4(\cos\theta)-\frac{1.1.3.5}{2.4.6.8}\frac{a^7}{r^7}P_6(\cos\theta)+\cdots\right]$$

if $r > a$, and

$$V = \frac{2M}{a}\left[1-\frac{r}{a}P_1(\cos\theta)\right.$$
$$\left.+\frac{1}{2}\frac{r^2}{a^2}P_2(\cos\theta)-\frac{1.1}{2.4}\frac{r^4}{a^4}P_4(\cos\theta)+\frac{1.1.3}{2.4.6}\frac{r^6}{a^6}P_6(\cos\theta)-\cdots\right]$$

if $r < a$ and $\theta < \frac{\pi}{2}$.

EXAMPLES.

1. The potential function due to a homogeneous hemisphere whose axis is taken as the polar axis, is

$$V = \frac{M}{a}\left[\frac{a}{r}+\frac{3.1}{2.4}\frac{a^2}{r^2}P_1(\cos\theta)-\frac{3.1.1}{2.4.6}\frac{a^4}{r^4}P_3(\cos\theta)+\frac{3.1.1.3}{2.4.6.8}\frac{a^6}{r^6}P_5(\cos\theta)-\cdots\right]$$

if $r > a$, and is

$$V = \frac{M}{a}\left[\frac{3}{2}+\frac{3}{2}\frac{r}{a}P_1(\cos\theta)+\frac{r^2}{a^2}P_2(\cos\theta)\right.$$
$$\left.+\frac{3.1}{2.4}\frac{r^3}{a^3}P_3(\cos\theta)-\frac{3.1.1}{2.4.6}\frac{r^5}{a^5}P_5(\cos\theta)+\cdots\right]$$

if $r < a$ and $\theta > \frac{\pi}{2}$.

2. The potential function due to a solid sphere whose density is proportional to the distance from a diametral plane is, at an external point,

$$V = \frac{8}{15}\frac{M}{a}\left[\frac{5.3}{2.4}\frac{a}{r}+\frac{5.3.1}{2.4.6}\frac{a^3}{r^3}P_2(\cos\theta)\right.$$
$$\left.-\frac{5.3.1.1}{2.4.6.8}\frac{a^5}{r^5}P_4(\cos\theta)+\frac{5.3.1.1.3}{2.4.6.8.10}\frac{a^7}{r^7}P_6(\cos\theta)-\cdots\right].$$

3. The potential function due to the homogeneous oblate spheroid generated by the rotation of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its minor axis is, at an external point,

$$V=\frac{3}{2}\frac{M}{(a^2-b^2)}\left[\frac{x^2+a^2-b^2}{2(a^2-b^2)^{\frac{1}{2}}}\left(\sin^{-1}\frac{(a^2-b^2+bx)}{a\sqrt{x^2+a^2-b^2}}\right.\right.$$
$$\left.\left.+\sin^{-1}\frac{(a^2-b^2-bx)}{a\sqrt{x^2+a^2-b^2}}\right)-x\right]$$

if the point is on the axis of the spheroid at a distance x from its centre.

$$V=\frac{3M}{(a^2-b^2)^{\frac{1}{2}}}\left[\frac{1}{1.3}\frac{(a^2-b^2)^{\frac{1}{2}}}{r}-\frac{1}{3.5}\frac{(a^2-b^2)^{\frac{3}{2}}}{r^3}P_2(\cos\theta)\right.$$
$$\left.+\frac{1}{5.7}\frac{(a^2-b^2)^{\frac{5}{2}}}{r^5}P_4(\cos\theta)-\cdots\right]$$

if $r>(a^2-b^2)^{\frac{1}{2}}$, and

$$V=\frac{3M}{(a^2-b^2)^{\frac{1}{2}}}\left[\frac{\pi}{4}-\frac{r}{(a^2-b^2)^{\frac{1}{2}}}P_1(\cos\theta)+\frac{\pi}{4}\frac{r^2}{(a^2-b^2)}P_2(\cos\theta)\right.$$
$$\left.-\frac{1}{1.3}\frac{r^3}{(a^2-b^2)^{\frac{3}{2}}}P_3(\cos\theta)+\frac{1}{3.5}\frac{r^5}{(a^2-b^2)^{\frac{5}{2}}}P_5(\cos\theta)-\cdots\right]$$

if $r<(a^2-b^2)^{\frac{1}{2}}$ and $\theta<\frac{\pi}{2}$.

4. The potential function due to the homogeneous prolate spheroid generated by the rotation of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its major axis is, at an external point,

$$V=\frac{3M}{(a^2-b^2)^{\frac{1}{2}}}\left[\frac{1}{1.3}\frac{(a^2-b^2)^{\frac{1}{2}}}{r}+\frac{1}{3.5}\frac{(a^2-b^2)^{\frac{3}{2}}}{r^3}P_2(\cos\theta)\right.$$
$$\left.+\frac{1}{5.7}\frac{(a^2-b^2)^{\frac{5}{2}}}{r^5}P_4(\cos\theta)+\cdots\right]$$

if $r>(a^2-b^2)^{\frac{1}{2}}$.

81. The method employed in the last three articles may be stated in general as follows:—Whenever in a problem involving the solving of the special form of Laplace's Equation

$$rD_r^2(rV)+\frac{1}{\sin\theta}D_\theta(\sin\theta\, D_\theta V)=0,$$

the value of V is given or can be found for all points on the axis of X and this value can be expressed as a sum or a series involving only whole powers positive or negative of the radius vector of the point, the solution for a point

not on the axis can be obtained by multiplying each term by the appropriate Zonal Harmonic, subject only to the condition that the result if a series must be convergent.

It will be shown in the next article that $P_m(\cos\theta)$ is never greater than one nor less than minus one. Hence the series in question will be convergent for all values of r for which the original series was *absolutely convergent.*

82. In addition to the form given in (1) Art. 77 for $P_m(x)$ other forms are often useful.

It ought to be possible to develop $P_m(\cos\theta)$, which may be regarded as a function of θ, into a Fourier's Series, and such a development may be obtained, though with much labor, by the methods of Chapter II.

The development in terms of cosines of multiples of θ may be obtained much more easily by the following device.

We have seen in Art. 75 that $P_m(\cos\theta)$ is the coefficient of the *mth* power of z in the development of $(1-2z\cos\theta+z^2)^{-\frac{1}{2}}$ in a power series, and that if mod $z<1$ $(1-2z\cos\theta+z^2)^{-\frac{1}{2}}$ can be developed into such a series. We know by the Theory of Functions that only one such series exists, so that the method by which we may choose to obtain the development will not affect the result.

$$(1-2z\cos\theta+z^2)^{-\frac{1}{2}} = (1-z(e^{\theta i}+e^{-\theta i})+z^2)^{-\frac{1}{2}}$$

$$= (1-ze^{\theta i})^{-\frac{1}{2}}(1-ze^{-\theta i})^{-\frac{1}{2}}.$$

$(1-ze^{\theta i})^{-\frac{1}{2}}$ may be developed into an absolutely convergent series if mod $z<1$, by the Binomial Theorem. We have

$$(1-ze^{\theta i})^{-\frac{1}{2}} = 1+\frac{1}{2}ze^{\theta i}+\frac{1.3}{2.4}z^2e^{2\theta i}+\frac{1.3.5}{2.4.6}z^3e^{3\theta i}+\frac{1.3.5.7}{2.4.6.8}z^4e^{4\theta i}+\cdots$$

$$(1-ze^{-\theta i})^{-\frac{1}{2}} = 1+\frac{1}{2}ze^{-\theta i}+\frac{1.3}{2.4}z^2e^{-2\theta i}+\frac{1.3.5}{2.4.6}z^3e^{-3\theta i}+\frac{1.3.5.7}{2.4.6.8}z^4e^{-4\theta i}+\cdots$$

The product of these series will give a development for $(1-2z\cos\theta+z^2)^{-\frac{1}{2}}$ in power series. The coefficient of z^m is easily picked out, and must be equal to $P_m(\cos\theta)$. We thus get

$$P_m(\cos\theta) = \frac{1.3.5.\cdots(2m-1)}{2.4.6-\cdots 2m}\Big[e^{m\theta i}+e^{-m\theta i}+\frac{1}{2}\cdot\frac{2m}{2m-1}(e^{(m-2)\theta i}+e^{-(m-2)\theta i})$$

$$+\frac{1.3}{2.4}\cdot\frac{2m(2m-2)}{(2m-1)(2m-3)}(e^{(m-4)\theta i}+e^{-(m-4)\theta i})+\cdots\Big]$$

$$P_m(\cos\theta) = \frac{1.3.5.\cdots(2m-1)}{2.4.6.\cdots 2m}\left[2\cos m\theta + 2\frac{1.m}{1.(2m-1)}\cos(m-2)\theta\right.$$
$$+2\frac{1.3\,m(m-1)}{1.2(2m-1)(2m-3)}\cos(m-4)\theta$$
$$\left.+2\frac{1.3.5}{1.2.3}\frac{m(m-1)(m-2)}{(2m-1)(2m-3)(2m-5)}\cos(m-6)\theta+\cdots\right]. \quad (1)$$

If m is odd the development runs down to $\cos\theta$; if m is even to $\cos(0)$, but in that case the coefficient of $\cos(0)$, that is, the constant term, will not contain the factor 2 which is common to all the other terms, but will be simply $\left[\frac{1.3.5\cdots(m-1)}{2.4.6.\cdots m}\right]^2$.

We write out the values of $P_m(\cos\theta)$ for a few values of m

$$\left.\begin{aligned}
P_0(\cos\theta) &= 1\\
P_1(\cos\theta) &= \cos\theta\\
P_2(\cos\theta) &= \frac{1}{4}(3\cos 2\theta+1)\\
P_3(\cos\theta) &= \frac{1}{8}(5\cos 3\theta+3\cos\theta)\\
P_4(\cos\theta) &= \frac{1}{64}(35\cos 4\theta+20\cos 2\theta+9)\\
P_5(\cos\theta) &= \frac{1}{128}[63\cos 5\theta+35\cos 3\theta+30\cos\theta]\\
P_6(\cos\theta) &= \frac{1}{512}[231\cos 6\theta+126\cos 4\theta+105\cos 2\theta+50]\\
P_7(\cos\theta) &= \frac{1}{1024}[429\cos 7\theta+231\cos 5\theta+189\cos 3\theta+175\cos\theta]\\
P_8(\cos\theta) &= \frac{1}{16384}[6435\cos 8\theta+3432\cos 6\theta+2772\cos 4\theta\\
&\qquad +2520\cos 2\theta+1225].
\end{aligned}\right\} \quad (2)$$

Since all the coefficients in the second member of (1) are positive, and since each cosine has unity for its maximum value it is clear that $P_m(\cos\theta)$ has its maximum value when $\theta=0$; but we have shown in Art. 76 that $P_m(1)=1$. Therefore $P_m(\cos\theta)$ is never greater than unity if θ is real. It is also easily seen from (1) that $P_m(\cos\theta)$ can never be less than -1.

83. $P_m(x)$ can be very simply expressed as a derivative. We hav

$$P_m(x)=\frac{(2m-1)(2m-3)\cdots 1}{m!}\left[x^m-\frac{m(m-1)}{2.(2m-1)}x^{m-2}\right.$$

$$\left.+\frac{m(m-1)(m-2)(m-3)}{2.4.(2m-1)(2m-3)}x^{m-4}-\right.$$

$$\int_0^x P_m(x)dx=\frac{(2m-1)(2m-3)\cdots 1}{(m+1)!}\left[x^{m+1}-\frac{(m+1)m}{2.(2m-1)}x^{m-1}\right.$$

$$\left.+\frac{(m+1)m(m-1)(m-2)}{2.4.(2m-1)(2m-3)}x^{m-3}\right.$$

$$\int_0^{x^2} P_m(x)dx^2=\int_0^x dx\int_0^x P_m(x)dx$$

$$=\frac{(2m-1)(2m-3)\cdots 1}{(m+2)!}\left[x^{m+2}-\frac{(m+2)(m+1)}{2.(2m-1)}x^{m}\right.$$

$$\left.+\frac{(m+2)(m+1)m(m-1)}{2.4.(2m-1)(2m-3)}x^{m-2}\right.$$

$$\int_0^{x^m} P_m(x)dx^m=\frac{(2m-1)(2m-3)\cdots 1}{(2m)!}\left[x^{2m}-\frac{2m(2m-1)}{2.(2m-1)}x^{2m-2}\right.$$

$$\left.+\frac{2m(2m-1)(2m-2)(2m-3)}{2.4.(2m-1)(2m-3)}x^{2m-4}-\right.$$

$$=\frac{(2m-1)(2m-3)\cdots 1}{(2m)!}\left[x^{2m}-mx^{2m-2}+\frac{m(m-1)}{2!}x^{2m-4}\right.$$

$$\left.-\frac{m(m-1)(m-2)}{3!}x^{2m-6}+\cdots\right.$$

The quantity in brackets obviously differs from $(x^2-1)^m$ by terms inv lower powers of x than the mth.

Hence
$$P_m(x)=\frac{1.3.5\cdots(2m-1)}{(2m)!}\frac{d^m}{dx^m}(x^2-1)^m,$$

or
$$P_m(x)=\frac{1}{2^m m!}\frac{d^m}{dx^m}(x^2-1)^m.$$

This important formula is entirely general and holds not merely $x=\cos\theta$, but for all values of x.

84. The last result is so important that it is worth while to confirm it by obtaining it directly from Legendre's Equation

$$(1-x^2)\frac{d^2z}{dx^2}-2x\frac{dz}{dx}+m(m+1)z=0 \tag{1}$$

v. (1) Art. 75.

Let us differentiate (1) with respect to x a few times representing $\frac{dz}{dx}$ by z', $\frac{d^2z}{dx^2}$ by z'', $\frac{d^3z}{dx^3}$ by z''', &c. We get

$$(1-x^2)\frac{d^2z'}{dx^2}-2.2x\frac{dz'}{dx}+[m(m+1)-2]z'=0,$$

$$(1-x^2)\frac{d^2z''}{dx^2}-2.3x\frac{dz''}{dx}+[m(m+1)-2(1+2)]z''=0,$$

$$(1-x^2)\frac{d^2z'''}{dx^2}-2.4x\frac{dz'''}{dx}+[m(m+1)-2(1+2+3)]z'''=0,$$

and in general

$$(1-x^2)\frac{d^2z^{(n)}}{dx^2}-2(n+1)x\frac{dz^{(n)}}{dx}+[m(m+1)-2(1+2+3+\cdots+n)]z^{(n)}=0$$

or $$(1-x^2)\frac{d^2z^{(n)}}{dx^2}-2(n+1)x\frac{dz^{(n)}}{dx}+[m(m+1)-n(n+1)]z^{(n)}=0. \tag{2}$$

Following the analogy of these steps it is easy to write equations that will differentiate into (1).

Let $\frac{dz_1}{dx}=z$, $\frac{d^2z_2}{dx^2}=z$, $\frac{d^3z_3}{dx^3}=z$, &c. Then

$$(1-x^2)\frac{d^2z_1}{dx^2}+m(m+1)z_1=0$$

will differentiate into (1),

$$(1-x^2)\frac{d^2z_2}{dx^2}+2.1x\frac{dz_2}{dx}+[m(m+1)-2.1]z_2=0$$

if differentiated twice will give (1),

$$(1-x^2)\frac{d^2z_3}{dx^2}+2.2x\frac{dz_3}{dx}+[m(m+1)-2(1+2)]z_3=0$$

if differentiated three times will give (1), and in general

$$(1-x^2)\frac{d^2z_n}{dx^2}+2(n-1)x\frac{dz_n}{dx}+[m(m+1)-n(n-1)]\,z_n=0 \tag{3}$$

if differentiated n times with respect to x will give (1).

If $n=m+1$ (3) reduces to

$$(1-x^2)\frac{d^2z_{m+1}}{dx^2}+2mx\frac{dz_{m+1}}{dx}=0, \tag{4}$$

and the $(m+1)$st derivative with respect to x of any function of x satisfies (4) will be a solution of (1). (4) can be written

$$(1-x^2)\frac{dz_m}{dx}+2mxz_m=0$$

and can be readily solved by separating the variables and integrating. Cal. (1) page 314. It gives

$$z_m=C(x^2-1)^m.$$

Hence
$$z=\frac{d^mz_m}{dx^m}=C\frac{d^m(x^2-1)^m}{dx^m}$$

is a solution of Legendre's Equation (1) and agrees with the value of obtained in Art. 83.

85. The equations obtained in Art. 84 are so curious and so simply that it is worth while to consider them a little more fully.

We have seen that

$$(1-x^2)\frac{d^2z}{dx^2}+2mx\frac{dz}{dx}=0$$

differentiates into

$$(1-x^2)\frac{d^2z}{dx^2}+2(m-1)x\frac{dz}{dx}+2mz=0;$$

that if we differentiate (2) m times we get Legendre's Equation

$$(1-x^2)\frac{d^2z}{dx^2}-2x\frac{dz}{dx}+m(m+1)z=0;$$

that if we differentiate (2) $2m$ times we get

$$(1-x^2)\frac{d^2z}{dx^2}-2(m+1)x\frac{dz}{dx}=0;$$

that if we differentiate (2) $m-n$ times we have

$$(1-x^2)\frac{d^2z}{dx^2}+2(n-1)x\frac{dz}{dx}+[m(m+1)-n(n-1)]z=0;$$

and that if we differentiate (2) $m+n$ times we have

$$(1-x^2)\frac{d^2z}{dx^2}-2(n+1)x\frac{dz}{dx}+[m(m+1)-n(n+1)]z=0.$$

By the aid of (1) we found in the last article a particular solution namely

$$z=(x^2-1)^m.$$

If we substitute in (2) $z = u(x^2-1)^m$ following the method illustrated fully in Art. 18, we get as the general solution of (2)

$$z = A(x^2-1)^m + B(x^2-1)^m \int \frac{dx}{(x^2-1)^{m+1}}, \tag{7}$$

A and B being arbitrary constants.

$\int \frac{dx}{(x^2-1)^{m+1}}$ is easily written out [v. formula (42) page 6. Table of Integrals. Int. Cal. Appendix]. If $x<1$ it vanishes when $x=0$. If $x>1$ it vanishes when $x=\infty$. If then $x<1$ (7) can be written

$$z = A(x^2-1)^m + B(x^2-1)^m \int_0^x \frac{dx}{(x^2-1)^{m+1}} \tag{8}$$

and if $x>1$

$$z = A(x^2-1)^m + B(x^2-1)^m \int_x^\infty \frac{dx}{(x^2-1)^{m+1}} \tag{9}$$

and in these forms unnecessary arbitrary constants are avoided.

From (7) we can get the general solutions of (3), (4), (5), and (6).

$$z = A\frac{d^m(x^2-1)^m}{dx^m} + B\frac{d^m}{dx^m}\left[(x^2-1)^m \int \frac{dx}{(x^2-1)^{m+1}}\right] \tag{10}$$

is the general solution of (3).

$$z = A\frac{d^{2m}(x^2-1)^m}{dx^{2m}} + B\frac{d^{2m}}{dx^{2m}}\left[(x^2-1)^m \int \frac{dx}{(x^2-1)^{m+1}}\right] \tag{11}$$

is the general solution of (4).

$$z = A\frac{d^{m-n}(x^2-1)^m}{dx^{m-n}} + B\frac{d^{m-n}}{dx^{m-n}}\left[(x^2-1)^m \int \frac{dx}{(x^2-1)^{m+1}}\right] \tag{12}$$

is the general solution of (5).

$$z = A\frac{d^{m+n}(x^2-1)^m}{dx^{m+n}} + B\frac{d^{m+n}}{dx^{m+n}}\left[(x^2-1)^m \int \frac{dx}{(x^2-1)^{m+1}}\right] \tag{13}$$

is the general solution of (6).

In each of these forms A and B are arbitrary constants and the integral is to be taken from 0 to x if $x<1$ and from x to ∞ if $x>1$.

Of course (10) must be identical with the forms already obtained in Arts. 16 and 18 as general solutions of Legendre's Equation.

Equation (4) is so simple that it can be solved directly, and we get its solution in the form

$$z = A_1 + B_1 \int \frac{dx}{(x^2-1)^{m+1}} \tag{14}$$

which must be equivalent to (11).

Comparing (14) with (7), the solution of (2), we see that every solutio
can be obtained from a solution of (2) by dividing the latter by $(x^2 -$
in other words that if we write (2)

$$(1-x^2)\frac{d^2z}{dx^2} + 2(m-1)x\frac{dz}{dx} + 2mz = 0,$$

and (4) as
$$(1-x^2)\frac{d^2z_1}{dx^2} - 2(m+1)x\frac{dz_1}{dx} = 0$$

$z = z_1(x^2-1)^m$; and the substitution of this value in (2) will give (
the substitution of $z_1 = \frac{z}{(x^2-1)^m}$ in (4) will give (2).

We have, then, two ways of obtaining (4) from (2); we may differenti
$2m$ times with respect to x, or we may replace z in (2) by $z_1(x^2-1)^m$.

If we use the first method we have seen that Legendre's Equatio
midway between (2) and (4). That is if we differentiate (2) m times
(3) and if we then differentiate (3) m times we get (4). Let us see
half-way equation in our second process is Legendre's Equation.

If
$$z = y(x^2-1)^{\frac{m}{2}}$$
and
$$y = z_1(x^2-1)^{\frac{m}{2}}$$
$$z = z_1(x^2-1)^m.$$

So that if in (2) we replace z by $y(x^2-1)^{\frac{m}{2}}$ and then repeat the op
on the resulting equation we shall get (4). Making the first substitut
find,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[m(m+1) - \frac{m^2}{1-x^2}\right]y = 0,$$

not Legendre's Equation but a somewhat more general form. Of co
solution is

$$y = A(x^2-1)^{\frac{m}{2}} + B(x^2-1)^{\frac{m}{2}}\int\frac{dx}{(x^2-1)^{m+1}}.$$

(2) and (4) are special forms of (5) and (6). Let us try the experim
substituting in (5) $z = y(1-x^2)^{\frac{n}{2}}$ and in (6) $z = \frac{y}{(1-x^2)^{\frac{n}{2}}}$. We fin
both substitutions give the same equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[m(m+1) - \frac{n^2}{1-x^2}\right]y = 0.$$

The solution of (17) can be obtained from either (12) or (13) and is

$$y=\frac{1}{(1-x^2)^{\frac{n}{2}}}\left\{A\frac{d^{m-n}(x^2-1)^m}{dx^{m-n}}+B\frac{d^{m-n}}{dx^{m-n}}\left[(x^2-1)^m\int\frac{dx}{(x^2-1)^{m+1}}\right]\right\} \quad (18)$$

or

$$y=(1-x^2)^{\frac{n}{2}}\left\{A_1\frac{d^{m+n}(x^2-1)^m}{dx^{m+n}}+B_1\frac{d^{m+n}}{dx^{m+n}}\left[(x^2-1)^m\int\frac{dx}{(x^2-1)^{m+1}}\right]\right\} \quad (19)$$

which of course must be equivalent.

86. In addition to the value of $P_m(x)$ given in (1) Art. 83 there is another important derivative form which we shall proceed to obtain. It is

$$P_m(\cos\theta)=\frac{(-1)^m}{m!}r^{m+1}D_x^m\left(\frac{1}{r}\right). \quad (1)$$

We have seen in Art. 75 that $\frac{1}{r}\frac{1}{\sqrt{1-2\frac{r_1}{r}\cos\theta+\frac{r_1^2}{r^2}}}$ can be developed into a convergent series if $r_1<r$ and that the $(m+1)$st term of that series is $\frac{P_m(\cos\theta)r_1^m}{r^{m+1}}$. Let us obtain this term by Taylor's Theorem.

$$\frac{1}{r}\frac{1}{\sqrt{1-2\frac{r_1}{r}\cos\theta+\frac{r_1^2}{r^2}}}=\frac{1}{\sqrt{r^2-2r_1r\cos\theta+r_1^2}}=\frac{1}{\sqrt{x^2+y^2+z^2-2xr_1+r_1^2}}$$

$$=\frac{1}{\sqrt{(x-r_1)^2+y^2+z^2}}.$$

Regarding this as a function of $(x-r_1)$ and developing according to powers of r_1 by Taylor's Theorem we get as the $(m+1)$st term

$$\frac{(-1)^m}{m!}r_1^mD_x^m\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right] \quad \text{or} \quad \frac{(-1)^m}{m!}r_1^mD_x^m\left(\frac{1}{r}\right).$$

Hence
$$\frac{P_m(\cos\theta)}{r^{m+1}}=\frac{(-1)^m}{m!}D_x^m\left(\frac{1}{r}\right).$$

87. We have now obtained four different forms for our *zonal harmonic*, a polynomial in x, an expression involving cosines of multiples of θ, a form involving an ordinary mth derivative with respect to x, and a form involving a partial mth derivative with respect to x. We shall now get a form due to Laplace, involving a definite integral.

$$\int_0^\pi\frac{d\phi}{a-b\cos\phi}=\frac{\pi}{(a^2-b^2)^{\frac{1}{2}}} \quad (1)$$

if $a^2>b^2$ [v. Int. Cal. page 68].

$\dfrac{1}{(1-2xz+z^2)^{\frac{1}{2}}}$ can be expressed in the form $\dfrac{1}{(a^2-b^2)^{\frac{1}{2}}}$ by taking a
and $b=z\sqrt{x^2-1}$ and no matter what value x may have z can be take
that a^2 will be greater than b^2. Then by (1)

$$\frac{1}{(1-2xz+z^2)^{\frac{1}{2}}}=\frac{1}{\pi}\int_0^\pi\frac{d\phi}{1-zx-z\sqrt{x^2-1}.\cos\phi}=\frac{1}{\pi}\int_0^\pi\frac{d\phi}{1-z(x+\sqrt{x^2-}}$$

$$=\frac{1}{\pi}\int_0^\pi[1+(x+\sqrt{x^2-1}.\cos\phi)z+(x+\sqrt{x^2-1}.$$

$$+(x+\sqrt{x^2-1}.\cos\phi)^3z^3+\cdots]d\phi$$

if z is taken so small that the modulus of $z(x+\sqrt{x^2-1}.\cos\phi)$ is less th
by Art. 77 (2) $P_m(x)$ is the coefficient of z^m in the development of $\dfrac{}{(1-2}$

hence $$P_m(x)=\frac{1}{\pi}\int_0^\pi[x+\sqrt{x^2-1}.\cos\phi]^m d\phi.$$

By replacing ϕ by $\pi-\phi$ in (2) we get

$$P_m(x)=\frac{1}{\pi}\int_0^\pi[x-\sqrt{x^2-1}.\cos\phi]^m d\phi.$$

$$\frac{1}{(1-2xz+z^2)^{\frac{1}{2}}}=\frac{1}{z}\,\frac{1}{\left(1-2x\dfrac{1}{z}+\dfrac{1}{z^2}\right)^{\frac{1}{2}}}$$ and if mod $\dfrac{1}{z}<1$ or in othe

mod $z>1$ $\dfrac{1}{\left(1-2x\dfrac{1}{z}+\dfrac{1}{z^2}\right)^{\frac{1}{2}}}$ can be developed into a convergent seri

ing powers of $\dfrac{1}{z}$, and the coefficient of $\left(\dfrac{1}{z}\right)^m$ will be $P_m(x)$; but thi
the coefficient of z^{-m-1} in the development of $\dfrac{1}{(1-2xz+z^2)^{\frac{1}{2}}}$ acc
descending powers of z, mod z being greater than 1.

If now we let $a=zx-1$ and $b=z\sqrt{x^2-1}$, $a^2-b^2=1-2x$
z may be taken so great that $a^2-b^2>0$. Then by (1)

$$\frac{1}{(1-2xz+z^2)^{\frac{1}{2}}}=\frac{1}{\pi}\int_0^\pi\frac{d\phi}{zx-1-z\sqrt{x^2-1}.\cos\phi}$$

$$=\frac{1}{\pi}\int_0^\pi\frac{d\phi}{z(x-\sqrt{x^2-1}.\cos\phi)\left[1-\dfrac{1}{z(x-\sqrt{x^2-1}.\cos\phi)}\right.}$$

$$=\frac{1}{\pi}\int_0^\pi\frac{1}{(x-\sqrt{x^2-1}.\cos\phi)}\left[z^{-1}+\frac{1}{(x-\sqrt{x^2-1}.\cos\phi)}\right.$$

$$+\frac{1}{(x-\sqrt{x^2-1}.\cos\phi)^2}z^{-3}$$

and the coefficient of z^{-m-1} is $\frac{1}{\pi}\int_0^\pi \frac{d\phi}{[x - \sqrt{x^2-1}.\cos\phi]^{m+1}}$.

Hence
$$P_m(x) = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{[x - \sqrt{x^2-1}.\cos\phi]^{m+1}}. \quad (4)$$

Replace ϕ by $\pi - \phi$ and we get

$$P_m(x) = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{[x + \sqrt{x^2-1}.\cos\phi]^{m+1}}. \quad (5)$$

88. In the problems in which we have already used *Zonal Harmonics* (v. Arts. 78–81) we have been able to start with the value of the Potential Function at any point on the axis of X, and it has been necessary to develop the expression for V on that axis in terms of ascending or descending powers of x. If, however, we start with the value of V in terms of θ for some given value of r, that is on the surface of some sphere, we must develop the function of θ in terms of *zonal harmonics* of $\cos\theta$ (v. Art. 10), and our problem becomes the following:—To develop a given function of $\cos\theta$ in terms of zonal harmonics of $\cos\theta$, or to develop a given function of x in terms of the functions $P_m(x)$, x lying between 1 and -1.

The problem resembles closely that of developing in a Fourier's series, which we have already considered at such length.

Let
$$f(x) = A_0P_0(x) + A_1P_1(x) + A_2P_2(x) + A_3P_3(x) + \cdots \quad (1)$$
for all values of x from -1 to 1 and let it be required to determine the coefficients.

If $f(x)$ is single-valued and has only finite discontinuities between $x = -1$ and $x = 1$ we may proceed as in Art. 19.

Let us take $n+1$ terms of (1) and attempt to determine the coefficients. Take $n+1$ values of x at equal intervals Δx between $x=-1$ and $x=1$ so that $(n+2)\Delta x = 2$; $f(-1+\Delta x)$, $f(-1+2\Delta x)$, $f(-1+3\Delta x)$, $\cdots$ $f[-1+(n+1)\Delta x]$ will be the corresponding values of $f(x)$. Substitute these values in (1) and we have

$$\left.\begin{aligned}
f(-1+\Delta x) &= A_0P_0(-1+\Delta x) + A_1P_1(-1+\Delta x)\\
&\qquad + A_2P_2(-1+\Delta x) + \cdots + A_nP_n(-1+\Delta x)\\
f(-1+2\Delta x) &= A_0P_0(-1+2\Delta x) + A_1P_1(-1+2\Delta x)\\
&\qquad + A_2P_2(-1+2\Delta x) + \cdots + A_nP_n(-1+2\Delta x)\\
&\vdots\\
f(1-\Delta x) &= A_0P_0(1-\Delta x) + A_1P_1(1-\Delta x) + A_2P_2(1-\Delta x) + \cdots\\
&\qquad + A_nP_n(1-\Delta x),
\end{aligned}\right\} \quad (2)$$

that is, $n+1$ equations from which in theory the $n+1$ coefficients $A_0, A_1, \cdots A_n$ can be determined.

Following the analogy of Art. 24 let us multiply the first equat $P_m(-1+\Delta x).\Delta x$, the second by $P_m(-1+2\Delta x).\Delta x$, the th $P_m(-1+3\Delta x).\Delta x$, &c., and add the equations. The first member resulting equation is

$$\sum_{k=1}^{k=n+1} f(-1+k\Delta x)P_m(-1+k\Delta x).\Delta x,$$

and the coefficient of any A as A_l in the second member is

$$\sum_{k=1}^{k=n+1} P_m(-1+k\Delta x)P_l(-1+k\Delta x).\Delta x.$$

If now n is indefinitely increased (3) approaches as its limiting value

$$\int_{-1}^{1} f(x)P_m(x)dx$$

and (4) approaches $$\int_{-1}^{1} P_m(x)P_l(x)dx.$$

We have now to find the value of the integral (6) or as we shal it for the sake of greater convenience

$$\int_{-1}^{1} P_m(x)P_n(x)dx.$$

89. $$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{1}{2^{m+n}m!\,n!}\int_{-1}^{1}\frac{d^m(x^2-1)^m}{dx^m}.\frac{d^n(x^2-1)^n}{dx^n}dx$$

by (1) Art. 83.

$$\int_{-1}^{1}\frac{d^m(x^2-1)^m}{dx^m}.\frac{d^n(x^2-1)^n}{dx^n}dx = \left[\frac{d^m(x^2-1)^m}{dx^m}\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\right]_{-1}^{1}$$
$$-\int_{-1}^{1}\frac{d^{m+1}(x^2-1)^m}{dx^{m+1}}.\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}dx$$

by *integration by parts*.

Now if $z = X(x^2-1)^n$

$$\frac{dz}{dx} = 2nxX(x^2-1)^{n-1} + (x^2-1)^n\frac{dX}{dx} = (x^2-1)^{n-1}\left[2nxX + (x^2-1)\frac{d}{d}\right.$$

Hence the pth derivative with respect to x of any function of x con $(x^2-1)^n$ as a factor will contain $(x^2-1)^{n-p}$ as a factor if $p<n$.

$\dfrac{d^{n-1}(x^2-1)^n}{dx^{n-1}}$, then, contains (x^2-1) as a factor and is zero when $x=1$ and when $x=-1$, so that (1) reduces to

$$\int_{-1}^{1}\frac{d^m(x^2-1)^m}{dx^m}\cdot\frac{d^n(x^2-1)^n}{dx^n}\,dx=-\int_{-1}^{1}\frac{d^{m+1}(x^2-1)^m}{dx^{m+1}}\cdot\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\,dx.$$

It follows that

$$\int_{-1}^{1}\frac{d^m(x^2-1)^m}{dx^m}\cdot\frac{d^n(x^2-1)^n}{dx^n}\,dx=(-1)^p\int_{-1}^{1}\frac{d^{m+p}(x^2-1)^m}{dx^{m+p}}\cdot\frac{d^{n-p}(x^2-1)^n}{dx^{n-p}}\,dx$$

$$=(-1)^p\int_{-1}^{1}\frac{d^{m-p}(x^2-1)^m}{dx^{m-p}}\cdot\frac{d^{n+p}(x^2-1)^n}{dx^{n+p}}\,dx. \quad (3)$$

If $m<n$ we get from (3)

$$\int_{-1}^{1}\frac{d^m(x^2-1)^m}{dx^m}\cdot\frac{d^n(x^2-1)^n}{dx^n}\,dx=(-1)^m\int_{-1}^{1}\frac{d^{2m}(x^2-1)^m}{dx^{2m}}\cdot\frac{d^{n-m}(x^2-1)^n}{dx^{n-m}}\,dx$$

$$=(-1)^m(2m)!\left[\frac{d^{n-m-1}(x^2-1)^n}{dx^{n-m-1}}\right]_{-1}^{1}=0,$$

since
$$\frac{d^{2m}(x^2-1)^m}{dx^{2m}}=(2m)!.$$

If $m>n$

$$\int_{-1}^{1}\frac{d^m(x^2-1)^m}{dx^m}\cdot\frac{d^n(x^2-1)^n}{dx^n}\,dx=(-1)^n\int_{-1}^{1}\frac{d^{m-n}(x^2-1)^m}{dx^{m-n}}\cdot\frac{d^{2n}(x^2-1)^n}{dx^{2n}}\,dx$$

$$=(-1)^n(2n)!\left[\frac{d^{m-n-1}(x^2-1)^m}{dx^{m-n-1}}\right]_{-1}^{1}=0.$$

If, then, m is not equal to n

$$\int_{-1}^{1}P_m(x)P_n(x)dx=0. \quad (4)$$

If $m=n$ we have to find $\displaystyle\int_{-1}^{1}[P_m(x)]^2dx$.

$$\int_{-1}^{1}[P_m(x)]^2dx=\frac{1}{2^{2m}(m!)^2}\int_{-1}^{1}\frac{d^m(x^2-1)^m}{dx^m}\cdot\frac{d^m(x^2-1)^m}{dx^m}\,dx.$$

$$\int_{-1}^{1}\frac{d^m(x^2-1)^m}{dx^m}\cdot\frac{d^m(x^2-1)^m}{dx^m}\,dx=(-1)^m\int_{-1}^{1}\frac{d^{2m}(x^2-1)^m}{dx^{2m}}\cdot(x^2-1)^m dx$$

by (3),
$$=(-1)^m(2m)!\int_{-1}^{1}(x^2-1)^m dx.$$

$$\int_{-1}^{1}(x^2-1)^m dx = \int_{-1}^{1}(x-1)^m(x+1)^m dx = -\frac{m}{m+1}\int_{-1}^{1}(x-1)^{m-1}(x+1)^{m+1}dx$$

$$= (-1)^m \frac{m!}{(m+1)(m+2)\cdots 2m}\int_{-1}^{1}(x+1)^{2m}dx$$

$$= (-1)^m \frac{2^{2m+1}m!}{(m+1)(m+2)\cdots(2m+1)}\cdot$$

Hence $$\int_{-1}^{1}[P_m(x)]^2 dx = \frac{1}{2^{2m}(m!)^2}\cdot\frac{(-1)^m(2m)!(-1)^m m!2^{2m+1}}{(m+1)(m+2)\cdots(2m+1)}$$

or $$\int_{-1}^{1}[P_m(x)]^2 dx = \frac{2}{2m+1}\cdot \qquad (5)$$

90. The solution of the problem in Art. 88 is now readily obtained, and we have

$$f(x) = A_0P_0(x) + A_1P_1(x) + A_2P_2(x) + \cdots \qquad (1)$$

where $$A_m = \frac{2m+1}{2}\int_{-1}^{1}f(x)P_m(x)dx\,. \qquad (2)$$

The function and the series are equal for all values of x from $x=-1$ to $x=1$, and $f(x)$ is subject to no conditions save those which would enable us to develop it in a Fourier's Series. [v. Chapter III.]

Of course (1) can be written

$$f(\cos\theta) = A_0P_0(\cos\theta) + A_1P_1(\cos\theta) + A_2P_2(\cos\theta) + \cdots$$

where $$A_m = \frac{2m+1}{2}\int_{-1}^{1}f(\cos\theta)P_m(\cos\theta)d(\cos\theta)$$

or if $f(\cos\theta) = F(\theta)$

$$F(\theta) = A_0P_0(\cos\theta) + A_1P_1(\cos\theta) + A_2P_2(\cos\theta) + \cdots \qquad (3)$$

where $$A_m = \frac{2m+1}{2}\int_{0}^{\pi}F(\theta)P_m(\cos\theta)\sin\theta.d\theta \qquad (4)$$

and the development holds good from $\theta=0$ to $\theta=\pi$.

If $f(x)$ is an even function, that is, if $f(-x)=f(x)$ (1) and (2) can be somewhat simplified. For in that case it can be easily shown (v. Art. 77) that

$$\int_{-1}^{1}f(x)P_{2k}(x)dx = 2\int_{0}^{1}f(x)P_{2k}(x)dx\,,$$

and that
$$\int_{-1}^{1} f(x)P_{2k+1}(x)dx = 0;$$

so that if $f(-x) = f(x)$

$$f(x) = A_0 P_0(x) + A_2 P_2(x) + A_4 P_4(x) + A_6 P_6(x) + \cdots \qquad (5)$$

where
$$A_{2k} = (4k+1)\int_0^1 f(x)P_{2k}(x)dx. \qquad (6)$$

If $f(x)$ is an odd function, that is, if $f(-x) = -f(x)$, it can be shown in like manner that

$$f(x) = A_1 P_1(x) + A_3 P_3(x) + A_5 P_5(x) + A_7 P_7(x) + \cdots \qquad (7)$$

where
$$A_{2k+1} = (4k+3)\int_0^1 f(x)P_{2k+1}(x)dx. \qquad (8)$$

If it is only necessary that the development should hold for $0 < x < 1$ any function may be expressed in form (5) or (7) at pleasure.

91. We can establish the fact that $\int_{-1}^{1} P_m(x)P_n(x)dx = 0$ by a more general method than that used in Art. 89.

Let X_m be any solution of Legendre's Equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dz}{dx}\right] + m(m+1)z = 0 \qquad \text{[v. (1) Art. 16].}$$

which with its first derivative with respect to x is finite, continuous, and single-valued for values of x between -1 and 1, -1 and 1 being included.

Then
$$\frac{d}{dx}\left[(1-x^2)\frac{dX_m}{dx}\right] + m(m+1)X_m = 0 \qquad (1)$$

and
$$\frac{d}{dx}\left[(1-x^2)\frac{dX_n}{dx}\right] + n(n+1)X_n = 0. \qquad (2)$$

Multiply (1) by X_n and (2) by X_m and subtract and integrate and we get

$$[m(m+1) - n(n+1)]\int_{-1}^{1} X_m X_n dx = \int_{-1}^{1} X_m \frac{d}{dx}\left[(1-x^2)\frac{dX_n}{dx}\right]dx$$
$$- \int_{-1}^{1} X_n \frac{d}{dx}\left[(1-x^2)\frac{dX_m}{dx}\right]dx.$$

Integrate by parts,

$$[m(m+1)-n(n+1)]\int_{-1}^{1} X_m X_n dx = \left[X_m(1-x^2)\frac{dX_n}{dx} - X_n(1-x^2)\frac{dX_m}{dx}\right.$$

$$-\int_{-1}^{1}(1-x^2)\frac{dX_n}{dx}\frac{dX_m}{dx}dx + \int_{-1}^{1}(1-x^2)\frac{dX_m}{dx}\frac{dX_n}{dx}\,dx$$

Whence
$$\int_{-1}^{1} X_m X_n dx = 0$$

unless $m = n$.

(3) gives at once the important formula

$$\int_x^1 X_m X_n dx = \frac{(1-x^2)\left[X_n\dfrac{dX_m}{dx} - X_m\dfrac{dX_n}{dx}\right]}{m(m+1)-n(n+1)}$$

from which come as special cases

$$\int_x^1 P_m(x)P_n(x)dx = \frac{(1-x^2)\left[P_n(x)\dfrac{dP_m(x)}{dx} - P_m(x)\dfrac{dP_n(x)}{dx}\right]}{m(m+1)-n(n+1)}$$

and since $P_0(x) = 1$

$$\int_x^1 P_m(x)dx = \frac{(1-x^2)\dfrac{dP_m(x)}{dx}}{m(m+1)},$$

unless $m = 0$.

EXAMPLES.

1. Show that $\displaystyle\int_0^1 P_m(x)dx = 0$ if m is even and is not zero.

$$= (-1)^{\frac{m-1}{2}}\frac{1}{m(m+1)}\frac{3.5.7.\cdots m}{2.4.6.\cdots(m-1)}$$

odd. v. Art. 91 (7) and Art. 77 (10).

2. Show that

$$\int_0^1 P_m(x)P_n(x)dx = 0 \quad \text{if } m \text{ and } n \text{ are both even or both odd.}$$

$$= (-1)^{\frac{m+n+1}{2}}\frac{m!\,n!}{2^{m+n-1}(m-n)(m+n+1)\left(\frac{m}{2}!\right)^2\left(\right.}$$

if m is even and n odd. v. Art. 91 (6) and Art. 77 (8), (9), and (10). c Strutt (Lord Rayleigh) Lond. Phil. Trans. 1870, page 579.

3. Show that $\displaystyle\int_0^1 [P_m(x)]^2 dx = \frac{1}{2m+1}$ v. Art. 89 (5).

92. Formula (4) Art. 91 can be obtained directly from Laplace's Equation by the aid of *Green's Theorem* (v. Peirce's Newt. Pot. Func. § 48).

Take the special form of *Green's Theorem* [(148) § 48 Peirce's Newt. Pot. Func.]

$$\iiint (U\nabla^2 V - V\nabla^2 U)dxdydz = \int (UD_n V - VD_n U)ds \tag{1}$$

where ∇^2 stands for $(D_x^2 + D_y^2 + D_z^2)$, D_n is the partial derivative along the external normal, and the left-hand member is the space-integral through the space bounded by any closed surface, and the right-hand member is the surface integral taken over the same surface. (v. Int. Cal. Chapter XIV.)

If U and V are solutions of Laplace's Equation $\nabla^2 V = \nabla^2 U = 0$ and (1) reduces to

$$\int (UD_n V - VD_n U)ds = 0. \tag{2}$$

Now $r^m X_m$ and $r^n X_n$ are solutions of Laplace's Equation if $x = \cos\theta$ (v. Art. 16).

If the unit sphere is taken as the bounding surface and $U = r^m X_m$ and $V = r^n X_n$ (1) and (2) will hold good.

$$D_n U = D_r(r^m X_m) = mr^{m-1}X_m,$$

$$D_n V = nr^{n-1}X_n,$$

$$ds = \sin\theta.d\theta d\phi,$$

and (2) becomes $$\int_0^{2\pi} d\phi \int_0^{\pi} (nX_m X_n - mX_m X_n)\sin\theta.d\theta = 0$$

or $$2\pi(n-m)\int_0^{\pi} X_m X_n \sin\theta.d\theta = 0. \tag{3}$$

Since $x = \cos\theta$, $\sin\theta.d\theta = -dx$ and (3) reduces to

$$\int_{-1}^{1} X_m X_n dx = 0^* \tag{4}$$

unless $m = n$.

93. We can now solve completely the problem of Art. 10 which was in that article carried to the point where it was only necessary to develop a certain function of θ in the form

$$A_0 P_0(\cos\theta) + A_1 P_1(\cos\theta) + A_2 P_2(\cos\theta) + \cdots$$

* It should be noted that this proof is no more general than that of the last article, for, in order that Green's Theorem should apply to $r^m X_m$, this function and its first derivatives must be finite continuous and single-valued within and on the surface of the unit sphere. (v. Peirce, Newt. Pot. Func. § 48.)

given that $\qquad f(\theta) = 1 \quad \text{from} \quad \theta = 0 \quad \text{to} \quad \theta = \frac{\pi}{2}$

and $\qquad f(\theta) = 0 \quad \text{from} \quad \theta = \frac{\pi}{2} \quad \text{to} \quad \theta = \pi.$

This amounts to the same thing as developing $F(x)$ into the series

$$F(x) = A_0 P_0(x) + A_1 P_1(x) + A_2 P_2(x) + A_3 P_3(x) + \cdots$$

where $\qquad F(x) = 0 \quad \text{from} \quad x = -1 \quad \text{to} \quad x = 0$

and $\qquad F(x) = 1 \quad \text{from} \quad x = 0 \quad \text{to} \quad x = 1.$

By Art. 90 (1) and (2)

$$A_0 = \frac{1}{2}\int_0^1 P_0(x)dx = \frac{1}{2}\int_0^1 dx = \frac{1}{2},$$

and any coefficient $\qquad A_m = \frac{2m+1}{2}\int_0^1 P_m(x)dx.$

By Art. 91, Ex. 1

$$\int_0^1 P_m(x)dx = 0 \quad \text{if } m \text{ is even}$$

$$= (-1)^{\frac{m-1}{2}} \frac{1}{m(m+1)} \frac{1.3.5.\cdots m}{2.4.6.\cdots(m-1)} \quad \text{if } m \text{ is odd.}$$

Hence $\qquad A_m = 0 \quad \text{if } m \text{ is even}$

$$= (-1)^{\frac{m-1}{2}} \frac{2m+1}{2m+2} \cdot \frac{1.3.5.\cdots(m-2)}{2.4.6.\cdots(m-1)} \quad \text{if } m \text{ is odd.}$$

Then $\qquad F(x) = \frac{1}{2} + \frac{3}{4}P_1(x) - \frac{7}{8}\cdot\frac{1}{2}P_3(x) + \frac{11}{12}\cdot\frac{1.3}{2.4}P_5(x) - \cdots$

and $\qquad u = \frac{1}{2} + \frac{3}{4}rP_1(\cos\theta) - \frac{7}{8}\cdot\frac{1}{2}r^3P_3(\cos\theta) + \frac{11}{12}\cdot\frac{1.3}{2.4}r^5P_5(\cos\theta) + \cdots$

for any point within the sphere.

94. If in a problem on the Potential Function the value of V is g every point of a spherical surface and has circular symmetry* about a di of that surface the value of V at any point in space can be obtained.

We have to solve Laplace's Equation in the form

$$rD_r^2(rV) + \frac{1}{\sin\theta}D_\theta(\sin\theta\, D_\theta V) = 0$$

* See note on page 12.

subject to the conditions

$$V = f(\theta) \quad \text{when} \quad r = a$$

$$V = 0 \qquad \text{"} \qquad r = \infty.$$

We have $\quad f(\theta) = A_0P_0(\cos\theta) + A_1P_1(\cos\theta) + A_2P_2(\cos\theta) + \cdots$

where $\quad A_m = \frac{2m+1}{2}\int_0^\pi f(\theta)P_m(\cos\theta)\sin\theta.d\theta. \qquad$ v. Art. 90 (4).

Hence

$$V = A_0 + A_1\left(\frac{r}{a}\right)P_1(\cos\theta) + A_2\left(\frac{r}{a}\right)^2P_2(\cos\theta) + A_3\left(\frac{r}{a}\right)^3P_3(\cos\theta) + \cdots \quad (2)$$

is the required solution for a point within the sphere, and

$$V = A_0\left(\frac{a}{r}\right) + A_1\left(\frac{a}{r}\right)^2P_1(\cos\theta) + A_2\left(\frac{a}{r}\right)^3P_2(\cos\theta) + A_3\left(\frac{a}{r}\right)^4P_3(\cos\theta) + \cdots \quad (3)$$

is the required solution for an external point.

EXAMPLES.

1. If on the surface of a sphere of radius c V is constant and equal to a show that $V = a$ for any point within the sphere and $V = \frac{ac}{r}$ for any external point.

2. Two equal thin hemispherical shells of radius c placed together to form a spherical surface are separated by a thin non-conducting layer. Charges of statical electricity are placed on the two hemispheres one of which is then found to be at potential a and the other at potential b. Find the value of the potential function at any point.

$$V = \frac{a+b}{2} + (b-a)\left[\frac{3}{4}\frac{r}{c}P_1(\cos\theta) - \frac{7}{8}\cdot\frac{1}{2}\frac{r^3}{c^3}P_3(\cos\theta)\right.$$
$$\left. + \frac{11}{12}\cdot\frac{1.3}{2.4}\frac{r^5}{c^5}P_5(\cos\theta) - \cdots\right]$$

for an internal point

$$V = \frac{a+b}{2}\cdot\frac{c}{r} + (b-a)\left[\frac{3}{4}\frac{c^2}{r^2}P_1(\cos\theta) - \frac{7}{8}\cdot\frac{1}{2}\frac{c^4}{r^4}P_3(\cos\theta)\right.$$
$$\left. + \frac{11}{12}\cdot\frac{1.3}{2.4}\frac{c^6}{r^6}P_5(\cos\theta) - \cdots\right]$$

for an external point.

3. If $V_1=f(\cos\theta)$ when $r=a$ and $V_1=0$ when $r=b$ show th
$a<r<b$

$$V_1=\sum_{m=0}^{m=\infty}A_m\left(\frac{b^{m+1}}{r^{m+1}}-\frac{r^m}{b^m}\right)\left(\frac{b^{m+1}}{a^{m+1}}-\frac{a^m}{b^m}\right)^{-1}P_m(\cos\theta)$$

where
$$A_m=\frac{2m+1}{2}\int_{-1}^{1}f(x)P_m(x)dx.$$

4. If $V_2=F(\cos\theta)$ when $r=b$ and $V_2=0$ when $r=a$ th
$a<r<b$

$$V_2=\sum_{m=0}^{m=\infty}B_m\left(\frac{r^m}{a^m}-\frac{a^{m+1}}{r^{m+1}}\right)\left(\frac{b^m}{a^m}-\frac{a^{m+1}}{b^{m+1}}\right)^{-1}P_m(\cos\theta)$$

where
$$B_m=\frac{2m+1}{2}\int_{-1}^{1}F(x)P_m(x)dx.$$

5. If the value of the potential function is given arbitrarily on the s
of a spherical shell but has circular symmetry * about a diameter $V=$
(v. Exs. 3 and 4).

6. Two concentric hollow spherical conductors are insulated and c
The inner one of radius a is at potential p, and the outer one of radiu
potential q. Find V for any point in space.

$$V=p \quad\text{if}\quad r<a,$$

$$V=\frac{pa}{b-a}\left(\frac{b}{r}-1\right)+\frac{qb}{b-a}\left(1-\frac{a}{r}\right)\quad\text{if}\quad a<r<b,$$

$$V=\frac{qb}{r}\quad\text{if}\quad r>b.$$

7. If $V=0$ on the base of a hemisphere and $V=f(\cos\theta)$ on the
surface, show that for a point within the hemisphere

$$V=\sum_{k=0}^{k=\infty}A_{2k+1}\left(\frac{r}{a}\right)^{2k+1}P_{2k+1}(\cos\theta)$$

where
$$A_{2k+1}=(4k+3)\int_0^1 f(x)P_{2k+1}(x)dx$$ [v. Art.

8. If the convex surface of a solid hemisphere of radius a is kep
constant temperature unity and the base at the constant temperat
show that after the permanent state of temperatures is set up the tem
of any internal point is

$$u=\frac{3}{2}\frac{r}{a}P_1(\cos\theta)-\frac{7}{4}\cdot\frac{1}{2}\frac{r^3}{a^3}P_3(\cos\theta)+\frac{11}{6}\cdot\frac{1.3}{2.4}\frac{r^5}{a^5}P_5(\cos\theta)-\cdots$$

* See note on page 12.

9. A sphere of radius a and with blackened surface is exposed to the direct rays of the sun in air at the temperature zero. Find the *stationary temperature* of any internal point.

Suggestion: $D_r u + hu - Mf(\theta) = 0$ when $r = a$.

Let $$u = \sum A_m \frac{r^m}{a^m} P_m(\cos\theta), \quad \text{and} \quad f(\theta) = \sum B_m P_m(\cos\theta).$$
Then we have

$$\sum m \frac{A_m}{a} P_m(\cos\theta) + h\sum A_m P_m(\cos\theta) - M\sum B_m P_m(\cos\theta) = 0,$$

whence $$A_m = \frac{MB_m}{h + \dfrac{m}{a}}.$$

Here $f(\theta) = \cos\theta$ if $0 < \theta < \frac{\pi}{2}$ and $f(\theta) = 0$ if $\frac{\pi}{2} < \theta < \pi$.

$$f(\theta) = \frac{1}{4} + \frac{1}{2}P_1(\cos\theta) + \frac{5}{16}P_2(\cos\theta) - \frac{3}{32}P_4(\cos\theta) + \cdots$$
$$+ (-1)^{k+1}\frac{(4k+1)(2k)!}{(4k+4)(2k-1)2^{2k}(k!)^2}P_{2k}(\cos\theta) + \cdots$$

v. Art. 91 Exs. (2) and (3). cf. J. W. Strutt (Lord Rayleigh), Lond. Phil. Trans. vol. 160, page 587.

95. The formulas of Art. 90 enable us to develop a given function of x in terms of *Zonal Surface Harmonics*, the development holding true for values of x between -1 and $+1$. If, however, we can show by outside considerations that a given function of x can be expressed in Zonal Surface Harmonics, the development holding true for all values of x, the formulas of Art. 90 will give us the development in question.

For example if n is a positive integer x^n can be expressed in terms of Zonal Surface Harmonics no matter what the value of x, and no Harmonic of higher order than n will enter. For the formulas giving the values of $P_1(x), P_2(x), \cdots P_n(x)$ (v. Art. 77) may be regarded as n algebraic equations of the first degree in terms of $x, x^2, x^3, \cdots x^n$ and $P_1(x), P_2(x), \cdots P_n(x)$.

From these equations the $n-1$ quantities $x, x^2, x^3, \cdots x^{n-1}$, can be eliminated, and there will result an equation of the first degree in x^n and $P_1(x)$, $P_2(x), \cdots P_n(x)$, which will enable us to express x^n in the form

$$A_0 + A_1P_1(x) + A_2P_2(x) + \cdots + A_nP_n(x),$$

no matter what the value of x, and we shall have the same formula when $-1 < x < 1$ as when $x > 1$ or $x < -1$.

Let us obtain this development. By Art. 90 (1) and (2)

$$x^n = A_0P_0(x) + A_1P_1(x) + A_2P_2(x) + \cdots$$

where
$$A_m = \frac{2m+1}{2}\int_{-1}^{1} x^n P_m(x)\,dx.$$

Then
$$A_m = \frac{2m+1}{2}\,\frac{1}{2^m m!}\int_{-1}^{1} x^n \frac{d^m(x^2-1)^m}{dx^m}\,dx \qquad \text{by (1) A}$$

By *integration by parts* we get

$$\int_{-1}^{1} x^n \frac{d^m(x^2-1)^m}{dx^m}\,dx = n(n-1)(n-2)\cdots(n-m+1)\int_{-1}^{1} x^{n-m}(1-x^2)^m d$$
$$\text{if} \quad m < n+1,$$
$$= 0 \quad \text{if} \quad m > n.$$

By *integration by parts* we readily obtain the reduction formula

$$\int_{-1}^{1} x^p(1-x^2)^q dx = \frac{2q}{p+1}\int_{-1}^{1} x^{p+2}(1-x^2)^{q-1}dx$$

$$\int_{-1}^{1} x^{n-m}(1-x^2)^m dx = \frac{2^m m!}{(n-m+1)(n-m+3)\cdots(n+m-1)}\int_{-1}^{1} x^{n+}$$

$$\int_{-1}^{1} x^{n+m}dx = \frac{2}{n+m+1} \quad \text{if} \quad n+m \quad \text{is even},$$
$$= 0 \quad \text{if} \quad n+m \quad \text{is odd}.$$

Hence
$$A_m = \frac{(2m+1)n(n-1)(n-2)\cdots(n-m+1)}{(n-m+1)(n-m+3)(n-m+5)\cdots(n+m+1)}$$
$$\text{if} \quad m < n+1 \quad \text{and} \quad m+n \quad \text{is even},$$
$$= 0 \quad \text{if} \quad m > n \quad \text{or if} \quad m+n \quad \text{is odd}.$$

Therefore

$$x^n = \frac{n!}{1.3.5\cdots(2n+1)}\Big[(2n+1)P_n(x) + (2n-3)\frac{(2n+1)}{2}P_{n-2}(x)$$
$$+ (2n-7)\frac{(2n+1)(2n-1)}{2.4}P_{n-4}(x)$$
$$+ (2n-11)\frac{(2n+1)(2n-1)(2n-3)}{2.4.6}P_{n-6}(x) + \cdots$$

the second member ending with the term $\frac{1}{n+1}P_0(x)$ if n is even an the term $\frac{3}{n+2}P_1(x)$ if n is odd.

For convenience of reference we write out a few powers of x.

$$\left.\begin{aligned}
x^0 &= 1 = P_0(x)\\
x &= P_1(x)\\
x^2 &= \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x)\\
x^3 &= \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)\\
x^4 &= \frac{8}{35} P_4(x) + \frac{4}{7} P_2(x) + \frac{1}{5} P_0(x)\\
x^5 &= \frac{8}{63} P_5(x) + \frac{4}{9} P_3(x) + \frac{3}{7} P_1(x)\\
x^6 &= \frac{16}{231} P_6(x) + \frac{24}{77} P_4(x) + \frac{10}{21} P_2(x) + \frac{1}{7} P_0(x)\\
x^7 &= \frac{16}{429} P_7(x) + \frac{8}{39} P_5(x) + \frac{14}{33} P_3(x) + \frac{1}{3} P_1(x)\\
x^8 &= \frac{128}{6435} P_8(x) + \frac{64}{495} P_6(x) + \frac{48}{143} P_4(x) + \frac{40}{99} P_2(x) + \frac{1}{9} P_0(x).
\end{aligned}\right\} \quad (5)$$

If a given function of x can be expressed as a *terminating power series* it can be developed into a Zonal Harmonic Series by the aid of (4). Given that

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

let
$$f(x) = B_0 + B_1 P_1(x) + B_2 P_2(x) + B_3 P_3(x) + \cdots;$$

then picking out carefully the coefficient of $P_m(x)$ we have

$$B_m = \frac{m!}{1.3.5.\cdots(2m-1)}\left[a_m + \frac{(m+1)(m+2)}{2.(2m+3)} a_{m+2} + \frac{(m+1)(m+2)(m+3)(m+4)}{2.4.(2m+3)(2m+5)} a_{m+4} + \cdots\right]. \quad (6)$$

96. The development of $\frac{dP_n(x)}{dx}$ is useful and is easily obtained.

Let
$$\frac{dP_n(x)}{dx} = A_0 P_0(x) + A_1 P_1(x) + A_2 P_2(x) + \cdots$$

Then
$$A_m = \frac{2m+1}{2}\int_{-1}^{1} P_m(x) \frac{dP_n(x)}{dx}\, dx \quad (1)$$

by Art. 90 (2);

$$\int_{-1}^{1} P_m(x) \frac{dP_n(x)}{dx}\, dx = \Big[P_m(x) P_n(x)\Big]_{x=-1}^{x=1} - \int_{-1}^{1} P_n(x) \frac{dP_m(x)}{dx}\, dx. \quad (2)$$

$$[P_m(x)P_n(x)]_{x=-1}^{x=1} = 0 \text{ if } m+n \text{ is even}$$
$$= 2 \text{ if } m+n \text{ is odd.}$$

Since $P_n(x)$ is an algebraic polynomial of the nth degree in x, $\frac{dP_n(x}{dx}$ algebraic polynomial of the $n-1$st degree in x. Therefore in (1) m than n; consequently $\frac{dP_m(x)}{dx}$ is an algebraic polynomial in x of lower than n and

$$\int_{-1}^{1} P_n(x) \frac{dP_m(x)}{dx} dx = 0 \qquad \text{by Art.}$$

We get then $\quad A_m = 2m+1$ if $m+n$ is odd and $m < n$,

$= 0$ if $m+n$ is even or $m > n-1$;

$$\frac{dP_n(x)}{dx} = (2n-1)P_{n-1}(x) + (2n-5)P_{n-3}(x) + (2n-9)P_{n-5}(x) +$$

the second member ending with the term $3P_1(x)$ if n is even and w term $P_0(x)$ if n is odd.

From (3) a number of simple formulas are readily obtained. For ex

$$\frac{dP_{n+1}(x)}{dx} - \frac{dP_{n-1}(x)}{dx} = (2n+1)P_n(x)$$

$$\int_x^1 P_n(x)dx = \frac{1}{2n+1}[P_{n-1}(x) - P_{n+1}(x)].$$

$$(2n+1)x\frac{dP_n(x)}{dx} = n\frac{dP_{n+1}(x)}{dx} + (n+1)\frac{dP_{n-1}(x)}{dx}$$

[v. (4) and Article 77 (12)].

$$(x^2-1)\frac{dP_n(x)}{dx} = nxP_n(x) - nP_{n-1}(x)$$

[v. (5) and Article 91 (7).

97. By the aid of the formulas of Art. 96 a number of valuable d ments can be obtained.

Let us get $\cos n\theta$ and $\sin n\theta$ n being any positive real.

$z = \cos n\theta$ and $z = \sin n\theta$ are solutions of the equation

$$\frac{d^2z}{d\theta^2} + n^2z = 0$$

or if we let $x = \cos\theta$, of the equation

$$(1-x^2)\frac{d^2z}{dx^2} - x\frac{dz}{dx} + n^2 z = 0. \tag{1}$$

Let $$a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + \cdots$$

be the required development of $\cos n\theta$ or of $\sin n\theta$.

Then $$\sum_{m=0}^{m=\infty} a_m\left[(1-x^2)\frac{d^2P_m(x)}{dx^2} - x\frac{dP_m(x)}{dx} + n^2P_m(x)\right] = 0 \qquad \text{by (1).}$$

$z = P_m(x)$ is a solution of Legendre's Equation (v. Art. 77). Hence

$$(1-x^2)\frac{d^2P_m(x)}{dx^2} - x\frac{dP_m(x)}{dx} = x\frac{dP_m(x)}{dx} - m(m+1)P_m(x),$$

and (1) becomes

$$\sum_{m=0}^{m=\infty} a_m\left[x\frac{dP_m(x)}{dx} + [n^2 - m(m+1)]P_m(x)\right] = 0. \tag{2}$$

Formulas (4) and (6) of Art. 96 enable us to throw (2) into the form

$$\sum_{m=0}^{m=\infty} a_m\left[\frac{n^2-m^2}{2m+1}\frac{dP_{m+1}(x)}{dx} - \frac{n^2-(m+1)^2}{2m+1}\frac{dP_{m-1}(x)}{dx}\right] = 0. \tag{3}$$

(3) must be identically true. Therefore the coefficient of $\dfrac{dP_{m+1}(x)}{dx}$ must equal zero, and we have

$$a_{m+2} = \frac{2m+5}{2m+1}\cdot\frac{n^2-m^2}{n^2-(m+3)^2}a_m. \tag{4}$$

If we are developing $\cos n\theta$

$$a_0 = \frac{1}{2}\int_0^\pi \cos n\theta \sin\theta . d\theta \qquad \text{by Art. 90 (4),}$$

$$= \frac{1}{4}\int_0^\pi [\sin(n+1)\theta - \sin(n-1)\theta]d\theta,$$

$$a_0 = -\frac{1}{2}\cdot\frac{1+\cos n\pi}{n^2-1}; \tag{5}$$

and $$a_1 = \frac{3}{2}\int_0^\pi \cos n\theta \cos\theta \sin\theta . d\theta \qquad \text{by Art. 90 (4),}$$

$$a_1 = -\frac{3}{2}\cdot\frac{1-\cos n\pi}{n^2-4}. \tag{6}$$

(4), (5), and (6) give us

$$\cos n\theta = -\frac{1+\cos n\pi}{2(n^2-1)}\left[P_0(\cos\theta) + 5\,\frac{n^2}{n^2-3^2}\,P_2(\cos\theta) + 9\,\frac{n^2(n^2-2^2)}{(n^2-3^2)(n^2-5^2)}\,P_4(\cos\theta) + \cdots\right]$$

$$-\frac{1-\cos n\pi}{2(n^2-2^2)}\left[3P_1(\cos\theta) + 7\,\frac{n^2-1^2}{n^2-4^2}\,P_3(\cos\theta) + 11\,\frac{(n^2-1^2)(n^2-3^2)}{(n^2-4^2)(n^2-6^2)}\,P_5(\cos\theta) + \cdots\right].$$

If n is a whole number $1+\cos n\pi$ or $1-\cos n\pi$ will vanish and the will end with the term involving $P_n(\cos\theta)$. For this case (7) may be rev

$$\cos n\theta = \frac{1}{2}\cdot\frac{2.4.6.\cdots 2n}{3.5.7.\cdots(2n+1)}\left[(2n+1)P_n(\cos\theta) + (2n-3)\,\frac{n^2-(n+1)^2}{n^2-(n-2)^2}\,P_{n-2}(\cos\theta) + (2n-7)\,\frac{[n^2-(n+1)^2][n^2-(n-1)^2]}{[n^2-(n-2)^2][n^2-(n-4)^2]}\,P_{n-4}(\cos\theta) + \cdots\right.$$

If we are developing $\sin n\theta$

$$a_0 = \frac{1}{2}\int_0^\pi \sin n\theta \sin\theta . d\theta = -\frac{1}{2}\cdot\frac{\sin n\pi}{n^2-1}.$$

$$a_1 = \frac{3}{2}\int_0^\pi \sin n\theta\cos\theta\sin\theta . d\theta = \frac{3}{2}\cdot\frac{\sin n\pi}{n^2-2^2}$$

$$\sin n\theta = -\frac{1}{2}\cdot\frac{\sin n\pi}{n^2-1}\left[P_0(\cos\theta) + 5\,\frac{n^2}{n^2-3^2}\,P_2(\cos\theta) + 9\,\frac{n^2(n^2-2^2)}{(n^2-3^2)(n^2-5^2)}\,P_4(\cos\theta) + \cdots\right]$$

$$+\frac{1}{2}\cdot\frac{\sin n\pi}{n^2-2^2}\left[3P_1(\cos\theta) + 7\,\frac{n^2-1^2}{n^2-4^2}\,P_3(\cos\theta) + 11\,\frac{(n^2-1^2)(n^2-3^2)}{(n^2-4^2)(n^2-6^2)}\,P_5(\cos\theta) + \cdots\right].$$

If n is a whole number $\sin n\pi = 0$, and all the terms of (9) vanish those involving $P_{n-1}(\cos\theta)$, $P_{n+1}(\cos\theta)$, $P_{n+3}(\cos\theta)$ &c., which becom terminate. For this case it is necessary to compute a_{n-1} independentl

We have

$$a_{n-1} = \frac{2n-1}{2}\int_0^\pi \sin n\theta P_{n-1}(\cos\theta)\sin\theta . d\theta$$

$$= \frac{2n-1}{4}\int_0^\pi [\cos(n-1)\theta - \cos(n+1)\theta] P_{n-1}(\cos\theta) d\theta.$$

Hence $$a_{n-1} = \frac{2n-1}{4}\cdot\frac{1.3.5.\cdots(2n-3)}{2.4.6.\cdots(2n-2)}\pi \qquad \text{[v. Art. 82 (1)]},$$

and

$$\sin n\theta = \frac{\pi}{4}\cdot\frac{1.3.\cdots(2n-3)}{2.4.\cdots(2n-2)}\Big[(2n-1)P_{n-1}(\cos\theta)$$

$$+(2n+3)\frac{n^2-(n-1)^2}{n^2-(n+2)^2}P_{n+1}(\cos\theta)$$

$$+(2n+7)\frac{[n^2-(n-1)^2][n^2-(n+1)^2]}{[n^2-(n+2)^2][n^2-(n+4)^2]}P_{n+3}(\cos\theta)+\cdots\Big]. \quad (10)$$

EXAMPLES.

1. Show that

$$\csc\theta = \frac{\pi}{2}\left[1+5\left(\frac{1}{2}\right)^2 P_2(\cos\theta)+9\left(\frac{1.3}{2.4}\right)^2 P_4(\cos\theta)+13\left(\frac{1.3.5}{2.4.6}\right)^2 P_6(\cos\theta)+\cdots\right]$$

whence

$$\frac{1}{\sqrt{1-x^2}} = \frac{\pi}{2}\left[1+5\left(\frac{1}{2}\right)^2 P_2(x)+9\left(\frac{1.3}{2.4}\right)^2 P_4(x)+13\left(\frac{1.3.5}{2.4.6}\right)^2 P_6(x)+\cdots\right]$$

[v. Art. 90 (4) and Art. 82].

2. Show that

$$\operatorname{ctn}\theta = \frac{\pi}{2}\left[3\left(\frac{1}{2}\right)P_1(\cos\theta)+7\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)^2 P_3(\cos\theta)+11\left(\frac{5}{6}\right)\left(\frac{1.3}{2.4}\right)^2 P_5(\cos\theta)+\cdots\right]$$

whence

$$\frac{x}{\sqrt{1-x^2}} = \frac{\pi}{2}\left[3\left(\frac{1}{2}\right)P_1(x)+7\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)^2 P_3(x)+11\left(\frac{5}{6}\right)\left(\frac{1.3}{2.4}\right)^2 P_5(x)+\cdots\right]$$

[v. Art. 90 (4) and Art. 82].

3. By integrating the result of Ex. 1 and simplifying by the aid of Art. 96 (5), obtain the development

$$\sin^{-1}x = \frac{\pi}{2}\left[3\left(\frac{1}{2}\right)^2 P_1(x)+7\left(\frac{1}{2.4}\right)^2 P_3(x)\right.$$

$$\left.+11\left(\frac{1.3}{2.4.6}\right)^2 P_5(x)+15\left(\frac{1.3.5}{2.4.6.8}\right)^2 P_7(x)+\cdots\right]$$

whence $\theta = \frac{\pi}{2}\left[P_0(\cos\theta) - 3\left(\frac{1}{2}\right)^2 P_1(\cos\theta) - 7\left(\frac{1}{2.4}\right)^2 P_3(\cos\theta)\right.$

$$\left. - 11\left(\frac{1.3}{2.4.6}\right)^2 P_5(\cos\theta) - \cdots\right].$$

4. By integrating the result of Ex. 2 and simplifying by the aid o
(5) obtain

$$\sqrt{1-x^2} = \frac{\pi}{2}\left[\frac{1}{2} - 5\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^2 P_2(x) - 9\left(\frac{3}{6}\right)\left(\frac{1}{2.4}\right)^2 P_4(x)\right.$$

$$\left. - 13\left(\frac{5}{8}\right)\left(\frac{1.3}{2.4.6}\right)^2 P_6(x) + \cdots\right]$$

whence

$$\sin\theta = \frac{\pi}{2}\left[\frac{1}{2} P_0(\cos\theta) - 5\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^2 P_2(\cos\theta) - 9\left(\frac{3}{6}\right)\left(\frac{1}{2.4}\right)^2 P_4(\cos\theta) \cdot\right.$$

To make clearer the analogy of development in Zonal Harmonic Se
development in Fourier's Series we give on page 185 a cut represe
first seven Surface Zonal Harmonics $P_1(\cos\theta)$, $P_2(\cos\theta)$, $\cdots P_7(\cos\theta$
are of course somewhat complicated Trigonometric curves resembling
$\cos\theta$, $\cos 2\theta$, $\cdots\cos 7\theta$; and on page 186, the first four successive
mations to the Zonal Harmonic Series

$$\frac{1}{2} + \frac{3}{4} P_1(\cos\theta) - \frac{7}{8}\cdot\frac{1}{2} P_3(\cos\theta) + \frac{11}{12}\cdot\frac{1.3}{2.4} P_5(\cos\theta) - \cdots$$

[v. (1) Art. 93], and

$$\frac{\pi}{2}\left[P_0(\cos\theta) - 3\left(\frac{1}{2}\right)^2 P_1(\cos\theta) - 7\left(\frac{1}{2.4}\right)^2 P_3(\cos\theta)\right.$$

$$- 11\left(\frac{1.3}{2.4.6}\right)^2 P_5(\cos\theta) -$$

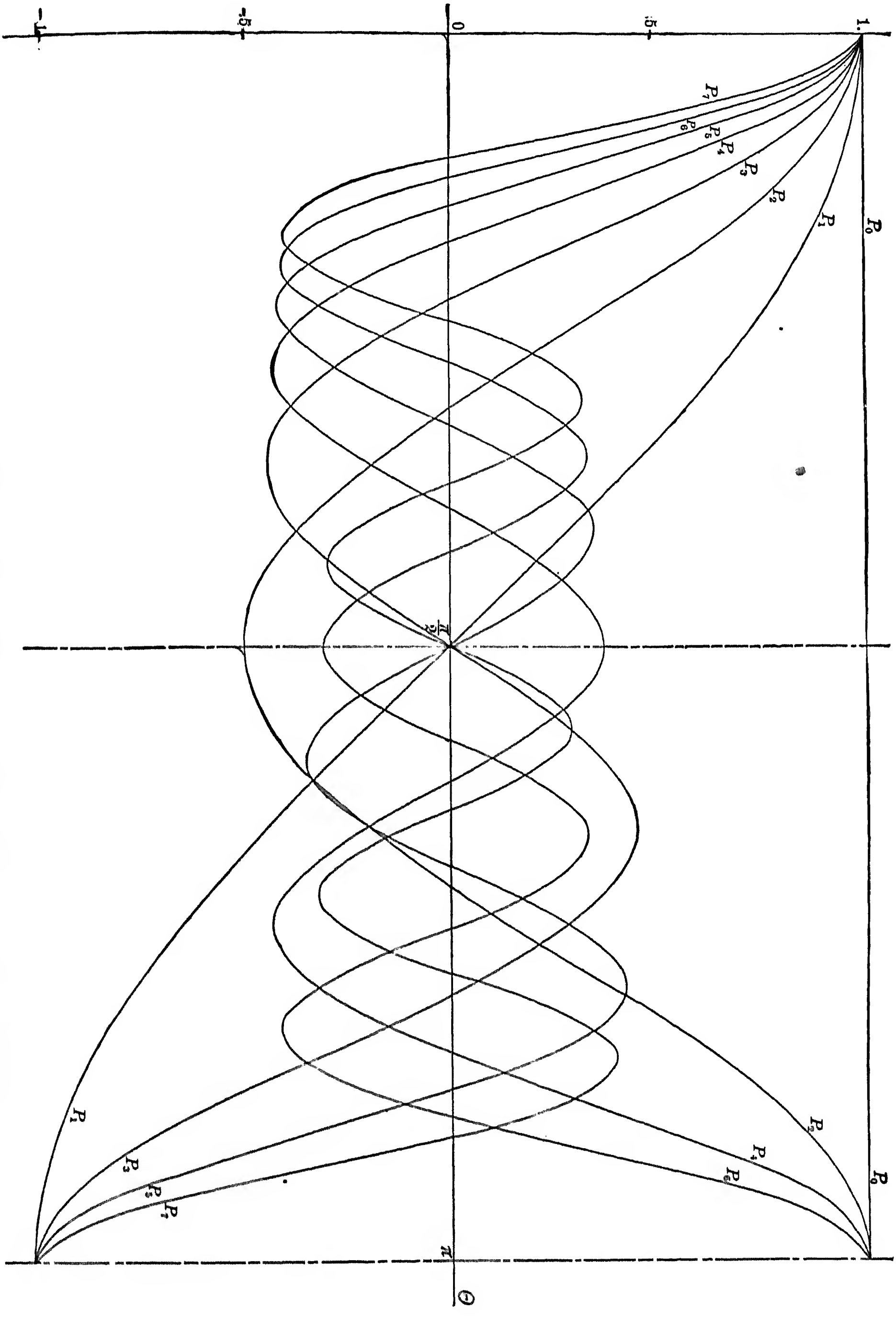
(v. Ex. 3 Art. 97).

[I] is equal to 1 from $\theta = 0$ to $\theta = \frac{\pi}{2}$, and to 0 from $\theta = \frac{\pi}{2}$ to $\theta =$
[II] is equal to θ from $\theta = 0$ to $\theta = \pi$.

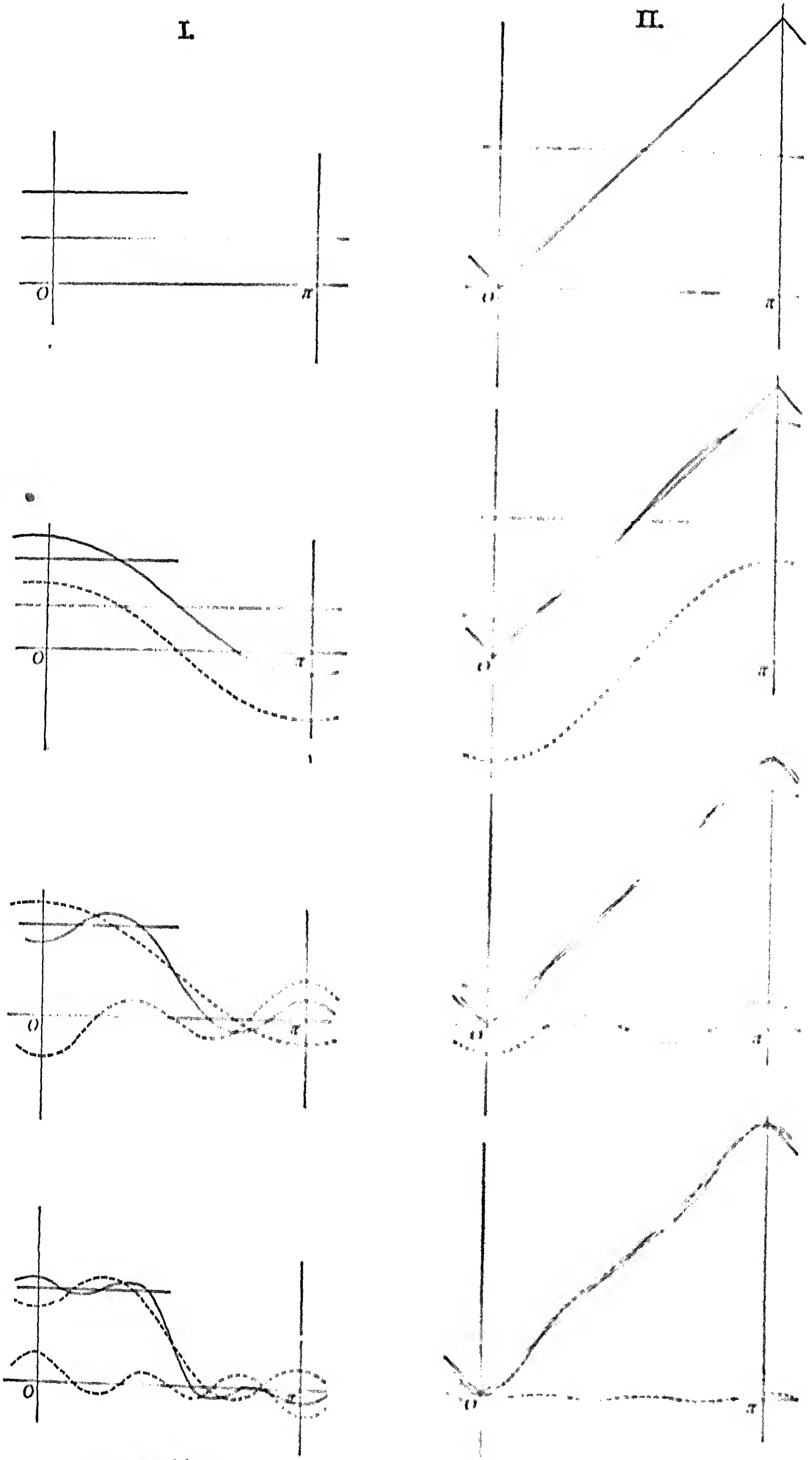
The figures on page 186 are constructed on precisely the same pri
those on pages 63 and 64, with which they should be carefully compa

98. By applying *Gauss's Theorem* (B. O. Peirce, Newt. Pot. Func.
the special Form of *Green's Theorem*,

$$\iiint \nabla^2 V dx\,dy\,dz = \int D_n V ds = -4\pi \iiint \rho\, dx\,dy\,dz,$$



The curves $y = P_0(\cos\theta)$, $y = P_1(\cos\theta)$, . . . $y = P_7(\cos\theta)$. (v. page 184.)



v. page 184.

[Peirce, N. P. F. § 49 (149)] to a box cut from an infinitely thin shell of attracting matter by a tube of force whose end is an element of the surface of the shell we readily obtain the important result

$$4\pi\rho\kappa = D_n V_1 - D_n V_2. \tag{1}$$

where ρ is the density and κ the thickness of the shell, V_1 the value of the potential function due to the shell at an internal point and V_2 its value at an external point, and where D_n is the partial derivative along the external normal to the outer surface of the shell.

If we have to deal with a surface distribution of matter we have only to replace $\rho\kappa$ in (1) by σ where σ is the surface density, whence

$$4\pi\sigma = D_n V_1 - D_n V_2 \tag{2}$$

(v. Peirce, N. P. F. §§ 45, 46, and 47).

Formulas (1) and (2) enable us to solve problems in attraction when we know the density of the attracting mass, and problems in Statical Electricity when we know the distribution of the charge, by methods analogous to that of Art. 94.

For example let us find the value of the potential function due to a thin material spherical shell of density ρ and radius a.

Since V must be a solution of Laplace's Equation and must be finite both when $r = 0$ and $r = \infty$ we have

$$V_1 = \sum A_m r^m P_m(\cos\theta)$$

$$V_2 = \sum B_m \frac{1}{r^{m+1}} P_m(\cos\theta).$$

V_1 and V_2 must approach the same limiting values as r approaches a. Hence

$$\frac{B_m}{a^{m+1}} = A_m a^m$$

or

$$B_m = A_m a^{2m+1}.$$

$$D_n V_1 = D_r V_1 = \sum m r^{m-1} A_m P_m(\cos\theta),$$

$$D_n V_2 = D_r V_2 = -\sum (m+1) \frac{A_m a^{2m+1}}{r^{m+2}} P_m(\cos\theta).$$

Therefore by (1)

$$4\pi\rho\kappa = \sum (2m+1) A_m a^{m-1} P_m(\cos\theta)$$

if κ is the thickness of the shell.

Let $$\rho = f(\cos\theta) = \sum C_m P_m(\cos\theta)$$

where $$C_m = \frac{2m+1}{2}\int_{-1}^{1} f(x) P_m(x)\,dx$$ by Art

Then $$4\pi\kappa C_m = (2m+1)A_m a^{m-1},$$

$$A_m = \frac{4\pi\kappa C_m}{(2m+1)a^{m-1}}, \quad\text{and}\quad B_m = \frac{4\pi\kappa}{2m+1} C_m a^{m+2},$$

and $$V_1 = 4\pi a\kappa \sum \frac{C_m}{2m+1}\frac{r^m}{a^m} P_m(\cos\theta),$$

and $$V_2 = 4\pi a\kappa \sum \frac{C_m}{2m+1}\frac{a^{m+1}}{r^{m+1}} P_m(\cos\theta).$$

99. We can now get the value of the potential function due to a s shell of finite thickness, provided that its density can be expressed as terms of the form $Cr^k P_m(\cos\theta)$.

Let a be the radius of the outer surface and b be the radius of th surface of the shell.

1st.—Let $\rho = Cr^k P_m(\cos\theta)$. Then for the shell of radius s and thic

$$V_1 = 4\pi s\,ds \frac{Cs^k}{2m+1}\frac{r^m}{s^m} P_m(\cos\theta)$$ by (3)

and $$V_2 = 4\pi s\,ds \frac{Cs^k}{2m+1}\frac{s^{m+1}}{r^{m+1}} P_m(\cos\theta)$$ by (4)

Then if $r < b$

$$V = \int_b^a V_1 = \frac{4\pi C}{(2m+1)}\frac{(a^{k-m+2} - b^{k-m+2})}{(k-m+2)} r^m P_m(\cos\theta),$$

if $r > a$

$$V = \int_b^a V_2 = \frac{4\pi C}{(2m+1)}\frac{(a^{k+m+3} - b^{k+m+3})}{(k+m+3)}\frac{P_m(\cos\theta)}{r^{m+1}},$$

and if $b < r < a$

$$V = \int_b^r V_2 + \int_r^a V_1 = \frac{4\pi C}{2m+1}\left[\frac{r^{k+m+3} - b^{k+m+3}}{(k+m+3)r^{m+1}} + \frac{a^{k-m+2} - r^{k-m+2}}{(k-m+2)} r^m\right] P_m(\cos\theta).$$

2d.—If $\rho = \sum C_m r^k P_m(\cos\theta)$ the solutions will consist of sums of t the forms given in (1), (2), and (3).

EXAMPLES.

1. If the shell is homogeneous

$$V = 2\pi\rho(a^2 - b^2) \text{ if } r < b,$$

$$V = \frac{4}{3}\pi\rho(a^3 - b^3)\frac{1}{r} = \frac{M}{r} \text{ if } r > a,$$

$$V = 2\pi\rho\left[a^2 - \frac{2b^3}{3r} - \frac{r^2}{3}\right] \text{ if } b < r < a.$$

2. If the density is any given function of the distance from the centre $V = \frac{M}{r}$ if $r > a$, and $V =$ a constant if $r < b$.

3. If the density at any point of a solid sphere is proportional to the square of the distance from a diametral plane

$$V = \frac{M}{a}\left[\frac{a}{r} + \frac{2}{7}\frac{a^3}{r^3}P_2(\cos\theta)\right] \text{ if } r > a.$$

4. If the density at any point of a solid sphere is proportional to its distance from a diametral plane

$$V = \frac{M}{a}\left[\frac{a}{r} + \frac{1}{6}\frac{a^3}{r^3}P_2(\cos\theta) - \frac{1.1}{6.8}\frac{a^5}{r^5}P_4(\cos\theta) + \frac{1.1.3}{6.8.10}\frac{a^7}{r^7}P_6(\cos\theta) - \cdots\right]$$

if $r > a$. Compare Ex. 2 Art. 80.

100. We have seen in Art. 18 (c) (3) that

$$Q_m(x) = CP_m(x)\int\frac{dx}{(1-x^2)[P_m(x)]^2}, \tag{1}$$

no constant term being understood with $\int\frac{dx}{(1-x^2)[P_m(x)]^2}$.

$\frac{1}{(1-x^2)[P_m(x)]^2}$ is a rational fraction and becomes infinite only for $x = 1$, $x = -1$, and for the roots of $P_m(x) = 0$, all of which are real and lie between -1 and 1, as can be proved by the aid of the relation $P_m(x) = \frac{1}{2^m m!}\frac{d^m(x^2-1)^m}{dx^m}$.

If $x^2 > 1$ $\int_x^\infty\frac{dx}{(1-x^2)[P_m(x)]^2}$ is finite and determinate and contains no constant term. Hence if $x^2 > 1$

$$Q_m(x) = -P_m(x)\int_x^\infty\frac{dx}{(1-x^2)[P_m(x)]^2} = P_m(x)\int_x^\infty\frac{dx}{(x^2-1)[P_m(x)]^2} \tag{2}$$

for the constant factor of $Q_m(x)$ has been chosen so that $C = -1$.

If $x^2 < 1$ the second member of (2) is not finite and determinate, and we are thrown back to the form (1), and C proves to be unity.

(1) gives us readily

$$Q_0(x) = \frac{1}{2}\log\frac{1+x}{1-x} \tag{3}$$

$$Q_1(x) = -1 + \frac{x}{2}\log\frac{1+x}{1-x} \tag{4}$$

if $x^2 < 1$.

(2) gives us

$$Q_0(x) = \frac{1}{2}\log\frac{x+1}{x-1} \tag{5}$$

$$Q_1(x) = -1 + \frac{x}{2}\log\frac{x+1}{x-1} \tag{6}$$

if $x^2 > 1$.

From Art. 85 (10) it follows that

$$Q_m(x) = C\frac{d^m}{dx^m}\left[(x^2-1)^m\int_0^x\frac{dx}{(x^2-1)^{m+1}}\right] \quad \text{if} \quad x^2<1,$$

$$= C\frac{d^m}{dx^m}\left[(x^2-1)^m\int_x^\infty\frac{dx}{(x^2-1)^{m+1}}\right] \quad \text{if} \quad x^2>1.$$

C can be determined and is equal to $\dfrac{(-1)^{m+1}2^m m!}{(2m)!}$ if $x^2<1$, and is equal to $\dfrac{(-1)^m 2^m m!}{(2m)!}$ if $x^2>1$.

Hence
$$Q_m(x) = \frac{(-1)^{m+1}2^m m!}{(2m)!}\frac{d^m}{dx^m}\left[(x^2-1)^m\int_0^x\frac{dx}{(x^2-1)^{m+1}}\right] \tag{7}$$
if $x^2<1$,

and
$$Q_m(x) = \frac{(-1)^m 2^m m!}{(2m)!}\frac{d^m}{dx^m}\left[(x^2-1)^m\int_x^\infty\frac{dx}{(x^2-1)^{m+1}}\right] \tag{8}$$
if $x^2>1$.

(7) and (8) give us for $Q_0(x)$ and $Q_1(x)$ the values already written in (3), (4), (5), and (6).

By the repeated application of the formula

$$(m+1)Q_{m+1}(x) - (2m+1)xQ_m(x) + mQ_{m-1}(x) = 0, \tag{9}$$

which may be obtained for the case where $x^2 < 1$ from Art. 16 (13) and (14), and for the case where $x^2 > 1$ from Art. 16 (9), any Surface Zonal Harmonic of the Second Kind can be obtained from $Q_0(x)$ and $Q_1(x)$ as given in (3), (4), (5), and (6).

Analogous formulas for $p_m(x)$ and $q_m(x)$ can be obtained without difficulty from Art. 16 (4) and (5). They are

$$(m+1)^2 q_{m+1}(x) - (2m+1)x p_m(x) - m^2 q_{m-1}(x) = 0 \tag{10}$$

and

$$p_{m+1}(x) + (2m+1)\, x q_m(x) - p_{m-1}(x) = 0 \tag{11}$$

and they hold good for any value of m.

EXAMPLES.

1. Confirm the values of $Q_0(x)$ and $Q_1(x)$ given in Art. 100 (3), (4), (5), and (6) by expanding them and comparing them with Art. 16 (13), (14), and (9).

2. If the value of V on the surface of a cone of revolution can be expressed in terms of whole powers positive or negative of r, V can be found for any point in space, cf. Art. 81.

If $V = \sum\left(A_m r^m + \dfrac{B_m}{r^{m+1}}\right)$ when $\theta = a$ then

$$V = \sum\left(A_m r^m + \frac{B_m}{r^{m+1}}\right)\frac{P_m(\cos\theta)}{P_m(\cos a)}.$$

3. If $V = \sum\left(A_m r^m + \dfrac{B_m}{r^{m+1}}\right)$ when $\theta = a$, and $V = 0$ when $\theta = \beta$,

$$V = \sum\left(A_m r^m + \frac{B_m}{r^{m+1}}\right)\left[\frac{Q_m(\cos\beta)P_m(\cos\theta) - P_m(\cos\beta)Q_m(\cos\theta)}{P_m(\cos a)Q_m(\cos\beta) - P_m(\cos\beta)Q_m(\cos a)}\right].$$

4. Find V for points corresponding to values of θ between a and β when V can be given in terms of whole powers of r for $\theta = a$ and for $\theta = \beta$.

5. Find by the method of Art. 16 solutions of Legendre's Equation of the form

$$z = {}_1P_m(x) = 1 + \frac{m(m+1)}{2}(x-1) + \frac{(m-1)m(m+1)(m+2)}{2^2(2!)^2}(x-1)^2$$
$$+ \frac{(m-2)(m-1)m(m+1)(m+2)(m+3)}{2^3(3!)^2}(x-1)^3 + \cdots,$$

$$z = {}_{-1}P_m(x) = 1 - \frac{m(m+1)}{2}(x+1) + \frac{(m-1)m(m+1)(m+2)}{2^2(2!)^2}(x+1)^2$$
$$+ \frac{(m-2)(m-1)m(m+1)(m+2)(m+3)}{2^3(3!)^2}(x+1)^3 + \cdots.$$

If m is a whole number, ${}_1P_m(x) = P_m(x)$ and ${}_{-1}P_m(x) = (-1)^m P_m(x)$. No matter what the value of m, ${}_1P_m(x)$ is absolutely convergent for $-1 < x < 3$, and ${}_{-1}P_m(x)$ is absolutely convergent for $-3 < x < 1$.

6. By the aid of (7) Art. 16 show that

$$V=\frac{1}{\sqrt{r}}\sin(n\log r)k_n(\cos\theta)\,,\qquad V=\frac{1}{\sqrt{r}}\sin(n\log r)l_n(\text{c}\ldots$$

$$V=\frac{1}{\sqrt{r}}\cos(n\log r)k_n(\cos\theta)\,,\qquad V=\frac{1}{\sqrt{r}}\cos(n\log r)l_n(\text{c}\ldots$$

are solutions of Laplace's Equation

$$rD_r^2(rV)+\frac{1}{\sin\theta}D_\theta(\sin\theta\, D_\theta V)=0,$$

$$k_n(x)=p_{-\frac{1}{2}+ni}(x)=1+\frac{n^2+\left(\frac{1}{2}\right)^2}{2!}x^2+\frac{\left[n^2+\left(\frac{1}{2}\right)^2\right]\left[n^2+\ldots\right.}{4!}\ldots$$

$$+\frac{\left[n^2+\left(\frac{1}{2}\right)^2\right]\left[n^2+\left(\frac{5}{2}\right)^2\right]\left[n^2+\left(\frac{9}{2}\right)^2\right]}{6!}\ldots$$

and

$$l_n(x)=-q_{-\frac{1}{2}+ni}(x)=x+\frac{n^2+\left(\frac{3}{2}\right)^2}{3!}x^3+\frac{\left[n^2+\left(\frac{3}{2}\right)^2\right]\left[n^2+\ldots\right.}{5!}\ldots$$

$$+\frac{\left[n^2+\left(\frac{3}{2}\right)^2\right]\left[n^2+\left(\frac{7}{2}\right)^2\right]\left[n^2+\left(\frac{11}{2}\right)^2\right.}{7!}\ldots$$

$k_n(x)$ and $l_n(x)$ are convergent if $x^2<1$, but are divergent if x^2 …

7. Show by the aid of Example 5 that

$$V=\frac{1}{\sqrt{r}}\sin(n\log r)K_n(\cos\theta)\,,\qquad V=\frac{1}{\sqrt{r}}\sin(n\log r)K_n(-\text{c}\ldots$$

$$V=\frac{1}{\sqrt{r}}\cos(n\log r)K_n(\cos\theta)\,,\qquad V=\frac{1}{\sqrt{r}}\cos(n\log r)K_n(-\text{c}\ldots$$

are solutions of $\quad rD_r^2(rV)+\frac{1}{\sin\theta}D_\theta(\sin\theta\, D_\theta V)=0$

if
$$K_n(x)={}_1P_{-\frac{1}{2}+ni}(x)=1-\frac{n^2+\left(\frac{1}{2}\right)^2}{2}(x-1)$$

$$+\frac{\left[n^2+\left(\frac{1}{2}\right)^2\right]\left[n^2+\left(\frac{3}{2}\right)^2\right]}{2^2(2!)^2}(x-1)^2$$

$$-\frac{\left[n^2+\left(\frac{1}{2}\right)^2\right]\left[n^2+\left(\frac{3}{2}\right)^2\right]\left[n^2+\left(\frac{5}{2}\right)^2\right]}{2^3(3!)^2}(x-1)^3$$

and

$$K_n(-x) = {}_{-1}P_{-\frac{1}{2}+ni}(x) = 1 + \frac{n^2+\left(\frac{1}{2}\right)^2}{2}(x+1)$$
$$+\frac{\left[n^2+\left(\frac{1}{2}\right)^2\right]\left[n^2+\left(\frac{3}{2}\right)^2\right]}{2^2(2!)^2}(x+1)^2$$
$$+\frac{\left[n^2+\left(\frac{1}{2}\right)^2\right]\left[n^2+\left(\frac{3}{2}\right)^2\right]\left[n^2+\left(\frac{5}{2}\right)^2\right]}{2^3(3!)^2}(x+1)^3+\cdots.$$

$K_n(\cos\theta)$ is convergent except for $\theta=\pi$, and $K_n(-\cos\theta)$ is convergent except for $\theta=0$.

$k_n(x)$, $l_n(x)$, $K_n(x)$, and $K_n(-x)$ are sometimes called *Conal Harmonics*. They are particular values of z which satisfy Legendre's Equation written in the form

$$(1-x^2)\frac{d^2z}{dx^2}-2x\frac{dz}{dx}-\left(n^2+\frac{1}{4}\right)z=0.$$

For an elaborate treatment of them see E. W. Hobson on "A Class of Spherical Harmonics of Complex Degree." Trans. Camb. Phil. Soc., Vol. XIV.

8. If $V=f(r)$ when $\theta=\beta$,

$$V=\frac{1}{\pi\sqrt{r}}\int_{-\infty}^{\infty}d\lambda\int_0^{\infty}e^{\frac{\lambda}{2}}f(e^\lambda)\frac{K_\alpha(\cos\theta)}{K_\alpha(\cos\beta)}\cos\left[\alpha(\lambda-\log r)\right]d\alpha;\quad\text{if}\quad\theta<\beta.$$

9. If $V=f(r)$ when $\theta=\beta$ and $r<a$, and $V=0$ when $r=a$,

$$V=\frac{2}{\pi}\sqrt{\frac{a}{r}}\int_{-\infty}^{0}d\lambda\int_0^{\infty}e^{\frac{\lambda}{2}}f(ae^\lambda)\frac{K_\alpha(\cos\theta)}{K_\alpha(\cos\beta)}\sin\alpha\lambda\sin\left(\alpha\log\frac{r}{a}\right)d\alpha;\quad\text{if}\quad\theta<\beta.$$

10. If $V=f(r)$ when $\theta=\beta$ and $a<r<b$, and $V=0$ when $r=a$ and when $r=b$,

$$V=\sum_{m=1}^{m=\infty}A_m\frac{K_{m'}(\cos\theta)}{K_{m'}(\cos\beta)}\sin\left[\frac{m\pi(\log r-\log a)}{\log b-\log a}\right]$$

where
$$m'=\frac{m\pi}{\log b-\log a}$$
and

$$A_m=\frac{2}{\log b-\log a}\sqrt{\frac{a}{r}}\int_0^{\log\frac{b}{a}}e^{\frac{x}{2}}f(ae^x)\sin\frac{m\pi x}{\log b-\log a}dx;\quad\text{if}\quad\theta<\beta.$$

11. If $\theta > \beta$ $\cos\theta$ must be replaced by $(-\cos\theta)$ in examples 8,

12. If $V = f(r)$ when $\theta = \beta$, and $V = 0$ when $\theta = \gamma$,

$$V = \frac{1}{\pi\sqrt{r}}\int_{-\infty}^{\infty} d\lambda \int_0^{\infty} e^{\frac{\lambda}{2}} f(e^{\lambda}) \frac{k_a(\cos\theta)l_a(\cos\gamma) - k_a(\cos\gamma)l_a(\cos\theta)}{k_a(\cos\beta)l_a(\cos\gamma) - k_a(\cos\gamma)l_a(\cos\beta)} \cos[a(\lambda -$$

if $\beta < \theta < \gamma$.

13. If $V = f(r)$ when $\theta = \beta$ and $a < r < b$, $V = 0$ when θ
$a < r < b$, and $V = 0$ when $r = a$ and when $r = b$,

$$V = \sum_{m=1}^{m=\infty} A_m \frac{k_{m'}(\cos\theta)l_{m'}(\cos\gamma) - k_{m'}(\cos\gamma)l_{m'}(\cos\theta)}{k_{m'}(\cos\beta)l_{m'}(\cos\gamma) - k_{m'}(\cos\gamma)l_{m'}(\cos\beta)} \sin\frac{m\pi(\log r -}{\log b - \log}$$

where

$$m' = \frac{m\pi}{\log b - \log a}$$

$$A_m = \frac{2}{\log b - \log a}\sqrt{\frac{a}{r}}\int_0^{\log\frac{b}{a}} e^{\frac{x}{2}} f(ae^x) \sin\frac{m\pi x}{\log b - \log a}\, dx;$$

if $\beta < \theta < \gamma$ and $a < r < b$.

14. If $V = f(r)$ when $\theta = \beta$ and $a < r < b$, and $V = 0$ when
and $D_r V + hV = 0$ when $r = b$,

$$V = \sum_{m=1}^{m=\infty} A_m \frac{K_{a_m}(\cos\theta)}{K_{a_m}(\cos\beta)} \sin\left(a_m \log\frac{r}{a}\right).$$

$$A_m = \frac{2(a_m^2 + h^2b^2)}{a_m^2(\log b - \log a) + hb[hb(\log b - \log a) + 1]}\int_0^{\log\frac{b}{a}} e^{\frac{x}{2}} f(ae^x) \sin a_m$$

and a_m is a root of the equation

$$a\cos\left(a\log\frac{b}{a}\right) + hb\sin\left(a\log\frac{b}{a}\right) = 0 \qquad \text{v. Art.}$$

CHAPTER VI.

SPHERICAL HARMONICS.

101. When we are dealing with problems in finding the *potential function* due to forces which have not circular symmetry * about an axis and are using Spherical Coördinates, we have to solve Laplace's Equation in the form

$$rD_r^2(rV) + \frac{1}{\sin\theta} D_\theta(\sin\theta\, D_\theta V) + \frac{1}{\sin^2\theta} D_\phi^2 V = 0 \tag{1}$$

[v. (XIII) Art. 1].

To get a particular solution of (1) we shall assume as usual that V is a product of functions each of which involves but a single variable.

Let $V = R.\Theta.\Phi$; where R involves r only, Θ involves θ only, and Φ ϕ only. Substitute in (1) and we get

$$\frac{r}{R}\frac{d^2(rR)}{dr^2} + \frac{1}{\Theta\sin\theta}\frac{d\left(\sin\theta\frac{d\Theta}{d\theta}\right)}{d\theta} + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0 \tag{2}$$

or

$$\frac{r\sin^2\theta}{R}\frac{d^2(rR)}{dr^2} + \frac{\sin\theta}{\Theta}\frac{d\left(\sin\theta\frac{d\Theta}{d\theta}\right)}{d\theta} = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}.$$

As the first member does not contain ϕ the second member cannot contain ϕ, and as it contains no other variable it must be constant; call it n^2. Equation (2) is then equivalent to the two equations

$$\frac{d^2\Phi}{d\phi^2} + n^2\Phi = 0 \tag{3}$$

and

$$\frac{r}{R}\frac{d^2(rR)}{dr^2} + \frac{1}{\Theta\sin\theta}\frac{d\left[\sin\theta\frac{d\Theta}{d\theta}\right]}{d\theta} - \frac{n^2}{\sin^2\theta} = 0 \tag{4}$$

(3) has been solved before and gives us

$$\Phi = A\cos n\phi + B\sin n\phi \tag{5}$$

[v. Art. 13(*a*)].

The first term of (4) does not involve θ and the second and third terms do not involve r.

* See note, page 12.

$\frac{r}{R}\frac{d^2(rR)}{dr^2}$ must, then, be a constant; we shall call it $m(m+1)$ as in Art. 13(c). Then (4) breaks up into

$$r\frac{d^2(rR)}{dr^2} = m(m+1)R \tag{6}$$

and
$$\frac{1}{\sin\theta}\frac{d\left[\sin\theta\frac{d\Theta}{d\theta}\right]}{d\theta} + \left[m(m+1) - \frac{n^2}{\sin^2\theta}\right]\Theta = 0. \tag{7}$$

(6) was solved in Art. 13(c) and gives

$$R = A_1 r^m + B_1 r^{-m-1}. \tag{8}$$

If in (7) we replace $\cos\theta$ by μ we get

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + \left[m(m+1) - \frac{n^2}{1-\mu^2}\right]\Theta = 0, \tag{9}$$

the equivalent of

$$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + \left[m(m+1) - \frac{n^2}{1-x^2}\right]z = 0 \tag{10}$$

[v. (17) Art. 85], which was solved in Art. 85 for the case where m and n are positive integers and $n < m+1$. v. (18) and (19) Art. 85.

From (19) Art. 85 we get as a particular solution of (9)

$$\Theta = (1-\mu^2)^{\frac{n}{2}}\frac{d^n P_m(\mu)}{d\mu^n} = \sin^n\theta\frac{d^n P_m(\mu)}{d\mu^n}, \tag{11}$$

if we restrict ourselves to whole positive values of m and n, as we shall do hereafter unless the contrary is explicitly stated, and suppose m not less than n.

A second but less useful particular solution of (9) is

$$\Theta = (1-\mu^2)^{\frac{n}{2}}\frac{d^n Q_m(\mu)}{d\mu^n}.$$

Combining our results we have as important particular solutions of (1)

$$V = r^m(A\cos n\phi + B\sin n\phi)\sin^n\theta\frac{d^n P_m(\mu)}{d\mu^n}, \tag{12}$$

and
$$V = \frac{1}{r^{m+1}}(A\cos n\phi + B\sin n\phi)\sin^n\theta\frac{d^n P_m(\mu)}{d\mu^n}, \tag{13}$$

where m and n are positive integers and $n < m+1$.

102. $\sin^n\theta \dfrac{d^n P_m(\mu)}{d\mu^n}$ or $(1-\mu^2)^{\frac{n}{2}} \dfrac{d^n P_m(\mu)}{d\mu^n}$ is a new function of μ, that is of $\cos\theta$, and we shall represent it by $P_m^n(\mu)$* and shall call it an *associated function* of the nth order and mth degree. It is a value of Θ satisfying equation (9) Art 101.

By differentiating the value of $P_m(x)$ given in (9) Art. 74 we get the formula

$$P_m^n(\mu) = \frac{(2m)!\sin^n\theta}{2^m m!(m-n)!}\Big[\mu^{m-n} - \frac{(m-n)(m-n-1)}{2.(2m-1)}\mu^{m-n-2}$$
$$+ \frac{(m-n)(m-n-1)(m-n-2)(m-n-3)}{2.4.(2m-1)(2m-3)}\mu^{m-n-4} - \dots\Big] \quad (1)$$

the expression in the parenthesis ending with the term involving μ^0 if $m-n$ is even and with the term involving μ if $m-n$ is odd.

For convenience of reference we give on the next page a table from which $P_m^n(\mu)$ can be readily obtained for values of m and n from 1 to 8.

$\cos n\phi\, P_m^n(\mu)$ and $\sin n\phi\, P_m^n(\mu)$, that is,

$$\cos n\phi \sin^n\theta \frac{d^n P_m(\mu)}{d\mu^n} \quad \text{and} \quad \sin n\phi \sin^n\theta \frac{d^n P_m(\mu)}{d\mu^n}$$

are called *Tesseral Harmonics* of the mth degree and nth order, and are values of V which satisfy the equation

$$m(m+1)V + \frac{1}{\sin\theta} D_\theta(\sin\theta\, D_\theta V) + \frac{1}{\sin^2\theta} D_\phi^2 V = 0 \quad (2)$$

or its equivalent

$$m(m+1)V + D_\mu[(1-\mu^2)D_\mu V] + \frac{1}{1-\mu^2} D_\phi^2 V = 0. \quad (3)$$

There are obviously $2m+1$ Tesseral Harmonics of the mth degree, namely

$$P_m(\mu), \quad \cos\phi\sin\theta\frac{dP_m(\mu)}{d\mu}, \quad \sin\phi\sin\theta\frac{dP_m(\mu)}{d\mu}$$
$$\cos 2\phi\sin^2\theta\frac{d^2P_m(\mu)}{d\mu^2}, \quad \sin 2\phi\sin^2\theta\frac{d^2P_m(\mu)}{d\mu^2}$$
$$\cos 3\phi\sin^3\theta\frac{d^3P_m(\mu)}{d\mu^3}, \quad \sin 3\phi\sin^3\theta\frac{d^3P_m(\mu)}{d\mu^3}$$
$$\dots\dots\dots\dots$$
$$\cos m\phi\sin^m\theta\frac{d^mP_m(\mu)}{d\mu^m}, \quad \sin m\phi\sin^m\theta\frac{d^mP_m(\mu)}{d\mu^m}.$$

If each of these is multiplied by a constant and their sum taken, this sum is called a *Surface Spherical Harmonic* of the mth degree, and is a solution of equations (2) and (3). We shall represent it by $Y_m(\mu, \phi)$ or by $Y_m(\theta, \phi)$.

* Most of the English writers represent this function by $T_m^n(\mu)$.

m	$n = 1.$	$n = 2.$	$n = 3.$
1	1		
2	3μ	3	
3	$\frac{3}{2}(5\mu^2 - 1)$	15μ	15
4	$\frac{5}{2}(7\mu^3 - 3\mu)$	$\frac{15}{2}(7\mu^2 - 1)$	105μ
5	$\frac{15}{8}(21\mu^4 - 14\mu^2 + 1)$	$\frac{105}{2}(3\mu^3 - \mu)$	$\frac{105}{2}(9\mu^2$ …
6	$\frac{21}{8}(33\mu^5 - 30\mu^3 + 5\mu)$	$\frac{105}{8}(33\mu^4 - 18\mu^2 + 1)$	$\frac{315}{2}(11\mu^3$ …
7	$\frac{7}{16}(429\mu^6 - 495\mu^4 + 135\mu^2 - 5)$	$\frac{63}{8}(143\mu^5 - 110\mu^3 + 15\mu)$	$\frac{315}{8}(143\mu^4$ …
8	$\frac{9}{16}(715\mu^7 - 1001\mu^5 + 385\mu^3 - 35\mu)$	$\frac{315}{16}(143\mu^6 - 143\mu^4 + 33\mu^2 - 1)$	$\frac{3465}{8}(39\mu^5$ …

$r^m Y_m(\mu, \phi)$ and $\frac{1}{r^{m+1}} Y_m(\mu, \phi)$ are called *Solid Spherical Harmon*… *m*th degree, and are solutions of Laplace's Equation (1) Art. 101.

To formulate:—

$$Y_m(\mu, \phi) = \sum_{n=0}^{n=m} \left[A_n \cos n\phi \sin^n \theta \frac{d^n P_m(\mu)}{d\mu^n} + B_n \sin n\phi \sin^n \theta \frac{d^n P_m(\mu)}{d\mu^n} \right.$$

or $$Y_m(\mu, \phi) = A_0 P_m(\mu) + \sum_{n=1}^{n=m} [A_n \cos n\phi P_m^n(\mu) + B_n \sin n\phi P_m^n(\mu)]$$

is a Surface Spherical Harmonic of the *m*th degree.

A Tesseral Harmonic is a special case of a Surface Spherical Harm… a Zonal Harmonic a special case of a Tesseral Harmonic; $P_m(\mu)$ I… Tesseral Harmonic of the zeroth order and the *m*th degree; it … written $P_m^0(\mu)$.

EXAMPLES.

1. Show that

$$(1 - x^2)\frac{d^2 z}{dx^2} - 2x\frac{dz}{dx} + \left[m(m+1) - \frac{n^2}{1 - x^2} \right] z = 0$$

reduces to

$$(1 - x^2)\frac{d^2 y}{dx^2} - 2(n+1)x\frac{dy}{dx} + [m(m+1) - n(n+1)]y = 0$$

if we substitute $(1 - x^2)^{\frac{n}{2}} y$ for z, even when m and n are unrestricte…

$\csc^n\theta\, P_m^n(\mu) = \dfrac{d^n P_m(\mu)}{d\mu^n}$.

$n=4$.	$n=5$.	$n=6$.	$n=7$.	$n=8$.
105				
945μ	945			
$\frac{945}{2}(11\mu^2-1)$	10395μ	10395		
$\frac{3465}{2}(13\mu^3-3\mu)$	$\frac{10395}{2}(13\mu^2-1)$	135135μ	135135	
$\frac{10395}{8}(65\mu^4-26\mu^2+1)$	$\frac{135135}{2}(5\mu^3-\mu)$	$\frac{135135}{2}(15\mu^2-1)$	2027025μ	2027025

2. Show that if in the second equation of Ex. 1 we let $y=\Sigma a_k x^k$ we get

$$a_{k+2} = -\frac{(m-n-k)(m+n+1+k)}{(k+1)(k+2)}\, a_k \qquad \text{(v. Art. 16)}$$

whence $z = p_m^n(x)$ and $z = q_m^n(x)$ are solutions of the first equation of Ex. 1, no matter what the values of m and n, if

$$p_m^n(x) = (1-x^2)^{\frac{n}{2}}\left[1 - \frac{(m-n)(m+n+1)}{2!}x^2 + \frac{(m-n)(m-n-2)(m+n+1)(m+n+3)}{4!}x^4 - \cdots\right]$$

and

$$q_m^n(x) = (1-x^2)^{\frac{n}{2}}\left[x - \frac{(m-n-1)(m+n+2)}{3!}x^3 + \frac{(m-n-1)(m-n-3)(m+n+2)(m+n+4)}{5!}x^5 - \cdots\right].$$

If $m-n$ is a positive integer, $p_m^n(x)$ or $q_m^n(x)$ will terminate with the term involving x^{m-n}, and in that case

$$z = (1-x^2)^{\frac{n}{2}}\left[x^{m-n} - \frac{(m-n)(m-n-1)}{2.(2m-1)}x^{m-n-2} + \frac{(m-n)(m-n-1)(m-n-2)(m-n-3)}{2.4.(2m-1)(2m-3)}x^{m-n-4} - \cdots\right],$$

the parenthesis ending with a term involving x^0 if $m-n$ is even and x if $m-n$ is odd, is a solution of the first equation of Ex. 1. If m and n are integers this value of z is $\dfrac{2^m m!(m-n)!}{(2m)!} P_m^n(x)$.

103. We have seen in the last chapter that in many problems it is important to be able to express a given function of $\cos\theta$, that is of μ, in terms of Zonal Harmonics of μ. So it is often desirable to express a given function of μ and ϕ in terms of Tesseral Harmonics of μ and ϕ.

If, for example, we are trying to find the *Potential Function* due to certain forces and have the value of the function given for some given value of r, that is, on the surface of some given sphere whose centre is at the origin of coördinates, of course the given value will be a function of θ and ϕ and if we can express it in terms of Spherical Harmonics of θ and ϕ we have only to multiply each term by the proper power of r to get the required solution of the problem. For we shall then have a value of V satisfying Laplace's Equation and reducing to the given function of θ and ϕ on the surface of the given sphere.

104. Suppose that we have a function of μ and ϕ given for all points on the unit sphere, that is, for all values of μ from -1 to 1 and for all values of ϕ from 0 to 2π, μ and ϕ being independent variables, and that we wish to express it in terms of Surface Spherical Harmonics.

Assume that

$$f(\mu,\phi)=\sum_{m=0}^{m=\infty}\Big[A_{0,m}P_m(\mu)+\sum_{n=1}^{n=m}\Big(A_{n,m}\cos n\phi P_m^n(\mu)+B_{n,m}\sin n\phi P_m^n(\mu)\Big)\Big]. \quad (1)$$

Let us consider first a finite case, and attempt to determine the coefficients so that

$$f(\mu,\phi)=\sum_{m=0}^{m=p}\Big[A_{0,m}P_m(\mu)+\sum_{n=1}^{n=m}\Big(A_{n,m}\cos n\phi P_m^n(\mu)+B_{n,m}\sin n\phi P_m^n(\mu)\Big)\Big] \quad (2)$$

shall hold good at as many points of the sphere as possible. The expression in brackets in the second member of (2) is a Surface Spherical Harmonic of the mth degree and contains $2m+1$ constant coefficients. The whole number of coefficients to be determined is then the sum of an Arithmetical Progression of $p+1$ terms the first term of which is 1 and the last is $2p+1$, and is therefore equal to $(p+1)^2$.

Let the interval from $\mu=-1$ to $\mu=1$ be divided into $p+2$ parts each of which is $\Delta\mu$ so that $(p+2)\Delta\mu=2$, and let the interval from $\phi=0$ to $\phi=2\pi$ be divided into $p+2$ parts each of which is $\Delta\phi$ so that $(p+2)\Delta\phi=2\pi$.

Then if we substitute in equation (2) in turn the values $(-1+\Delta\mu, \Delta\phi)$, $(-1+2\Delta\mu, \Delta\phi), \cdots [-1+(p+1)\Delta\mu, \Delta\phi]$; $(-1+\Delta\mu, 2\Delta\phi)$, $(-1+2\Delta\mu, 2\Delta\phi), \cdots [-1+(p+1)\Delta\mu, 2\Delta\phi]; \cdots [-1+\Delta\mu, (p+1)\Delta\phi]$, $[-1+2\Delta\mu, (p+1)\Delta\phi), \cdots [-1+(p+1)\Delta\mu, (p+1)\Delta\phi]$; since the first member in each case will be known we shall have $(p+1)^2$ equations of the first degree containing no unknown except the $(p+1)^2$ coefficients, and from them the coefficients can be determined. When they are substituted in equation (2) it will hold good at the $(p+1)^2$ points of the unit sphere where $p+1$ circles of latitude whose planes are equidistant intersect $p+1$ meridians which divide the equator into equal arcs. If now p is indefinitely increased the limiting values of the coefficients will be the coefficients in equation (1), and (1) will hold good all over the surface of the unit sphere.

To determine any particular constant we multiply each of our $(p+1)^2$ equations by $\Delta\mu\,\Delta\phi$ times the coefficient of the constant in question in that equation and add the equations and then investigate the limiting form approached by the resulting equation as p is indefinitely increased.

As p is indefinitely increased the summation in question will approach an integration; and since $d\mu\, d\phi = -\sin\theta . d\theta\, d\phi$ is the element of surface of the unit sphere, and as the limits -1 and 1 of μ correspond to π and 0 of θ the integration is a *surface integration* over the surface of the unit sphere.

In determining any coefficient as $A_{n,m}$ in (1) the first member of the limiting form of our resulting equation will be

$$\int_0^{2\pi} d\phi \int_{-1}^{1} f(\mu, \phi) \cos n\phi\, P_m^n(\mu) d\mu.$$

In the second member we shall come across terms of the forms

$$\int_0^{2\pi} d\phi \int_{-1}^{1} \sin l\phi \cos n\phi\, P_m^l(\mu) P_m^n(\mu) d\mu, \quad \int_0^{2\pi} d\phi \int_{-1}^{1} \cos l\phi \cos n\phi\, P_m^l(\mu) P_m^n(\mu) d\mu,$$

$$\int_0^{2\pi} d\phi \int_{-1}^{1} \sin n\phi \cos n\phi\, [P_m^n(\mu)]^2 d\mu, \quad \int_0^{2\pi} d\phi \int_{-1}^{1} \cos^2 n\phi\, [P_m^n(\mu)]^2 d\mu,$$

and other terms all of which come under the form

$$\int_0^{2\pi} d\phi \int_{-1}^{1} Y_l(\mu, \phi)\, Y_m(\mu, \phi) d\mu,$$

where $Y_m(\mu, \phi)$ and $Y_l(\mu, \phi)$ are Surface Spherical Harmonics of different degrees.

If we are determining a coefficient $B_{n,m}$ the only difference is that $\sin n\phi$ and $\cos n\phi$ will be interchanged in the forms just specified.